Further Pure Mathematics 1

A-Level Further Mathematics

This handout covers Topic 1: **Further Pure Mathematics 进阶纯数学** 1. It adds new algebra, matrices, polar coordinates, more vectors, and proof by induction.

Roots of polynomial equations

For a polynomial equation, the **roots 根** are linked to the **coefficients 系数**. For $ax^2 + bx + c = 0$ with roots α, β:

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

For a cubic $ax^3 + bx^2 + cx + d = 0$ with roots α, β, γ:

$$\sum \alpha = -\frac{b}{a}, \qquad \sum \alpha\beta = \frac{c}{a}, \qquad \alpha\beta\gamma = -\frac{d}{a}.$$

To find an equation whose roots are changed in a simple way, use a **substitution 代换** (for example, put $w = \alpha + 1$).

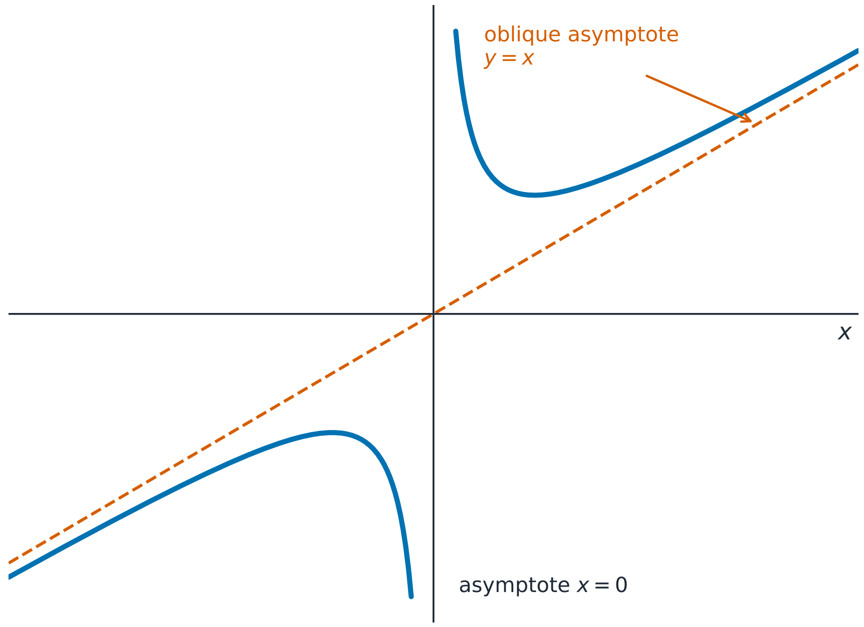
Worked example. The equation $x^2 - 5x + 6 = 0$ has roots α, β. Find the equation with roots $\alpha + 1, \beta + 1$.

Here $\alpha + \beta = 5$ and $\alpha\beta = 6$. The new sum is $(\alpha + 1) + (\beta + 1) = 7$, and the new product is $(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1 = 6 + 5 + 1 = 12$. So the new equation is

$$x^2 - 7x + 12 = 0.$$

Rational functions and graphs

A **rational function 有理函数** is a fraction of two polynomials. When the top has higher degree than the bottom, the graph has an **oblique asymptote 斜渐近线** (a slanted line the curve approaches); find it by dividing out. You should also be able to relate the graph of $y = f(x)$ to those of $y^2 = f(x)$, $y = \dfrac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$.



Far from the origin the curve hugs the slant line $y = x$; near $x = 0$ it runs off along the vertical asymptote.

Summation of series

Learn the standard results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

The **method of differences** 差分法 sums a series by cancelling middle terms: if each term is $f(r) - f(r+1)$, almost everything cancels. From the sum to n terms you can see whether a series is **convergent** 收敛 and, if so, find its **sum to infinity** 无穷和.

Worked example. Find $\sum_{r=1}^{n}(2r+1)$.

$$\sum_{r=1}^{n}(2r+1) = 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = 2 \cdot \tfrac{1}{2}n(n+1) + n = n(n+1) + n = n(n+2).$$

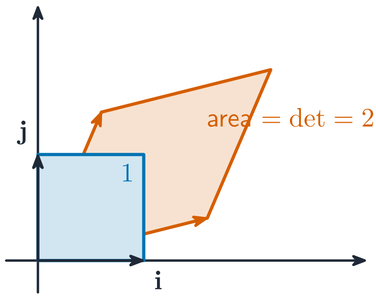
Matrices

A **matrix** 矩阵 is a rectangular block of numbers. You can add, subtract and multiply matrices (multiplication is row $\times$ column). The **zero matrix** 零矩阵 has every entry 0, and the **identity matrix** 单位矩阵 I leaves any matrix unchanged under multiplication.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the **determinant** 行列式 is $\det A = ad - bc$. If $\det A \neq 0$ the matrix is non-singular (otherwise it is a **singular matrix** 奇异矩阵), and the **inverse matrix** 逆矩阵 is

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For a product, $(AB)^{-1} = B^{-1}A^{-1}$. A 2×2 matrix can represent a **geometric transformation** 几何变换 of the plane: the determinant gives the area scale factor, and a product AB means "do B, then A". Points or lines that do not move are called **invariant points** 不变点 and **invariant lines** 不变直线.



The matrix sends the unit square to a parallelogram whose area is the determinant.

Worked example. Find the inverse of $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

$\det A = 3 \times 4 - 1 \times 2 = 10$, so

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}.$$

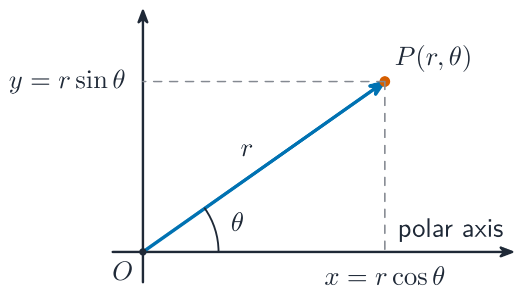
Polar coordinates

A spiral staircase: spirals are described naturally using polar coordinates.

Image: Colin, CC BY-SA 3.0 (commons.wikimedia.org)

Polar coordinates 极坐标 give a point by its distance r from the origin and its angle θ. They link to **Cartesian coordinates** 直角坐标 by

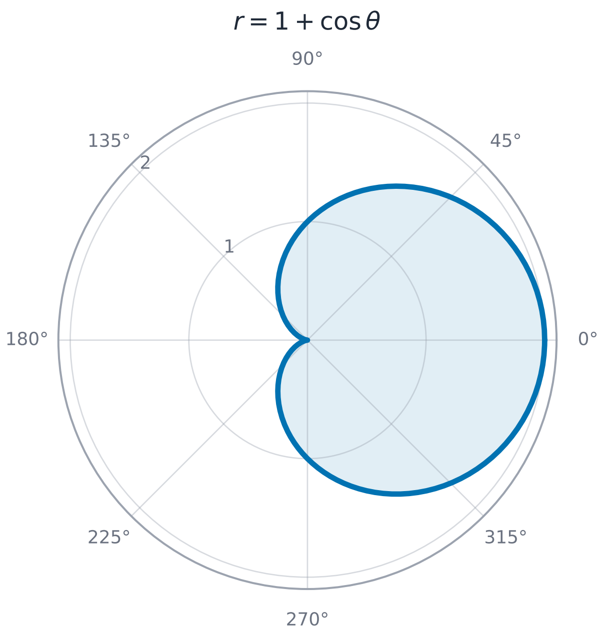
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2.$$



Polar coordinates (r,θ) convert to Cartesian by $x = r\cos\theta$ and $y = r\sin\theta$.

You should sketch simple **polar curves** 极坐标曲线, and find the area of a sector with

$$\text{area} = \tfrac{1}{2}\int r^2\, d\theta.$$



The cardioid $r = 1 + \cos\theta$, plotted straight from r as a function of θ.

Worked example. Convert the polar equation $r = 4\cos\theta$ to Cartesian form.

Multiply both sides by r: $r^2 = 4r\cos\theta$, so $x^2 + y^2 = 4x$. Completing the square gives $(x-2)^2 + y^2 = 4$, a circle of radius 2 centred at $(2, 0)$.

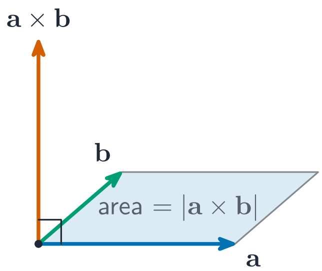
Vectors

Forces like wind and water are vectors —they have both size and direction.

In three dimensions a **plane** 平面 can be written as $ax + by + cz = d$, or $\mathbf{r} \cdot \mathbf{n} = p$, or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. Besides the scalar product, there is the **vector product** 向量积 of two **vectors** 向量:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{n}},$$

which gives a vector at right angles to both. With scalar and vector products you can find distances, angles, and where lines and planes meet —including the shortest distance between **skew lines** 异面直线.



$\mathbf{a} \times \mathbf{b}$ is perpendicular to both vectors; its length equals the area of the parallelogram they span.

Worked example. Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}(1) - \mathbf{j}(1) + \mathbf{k}(1) = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Proof by induction

Mathematical induction 数学归纳法 proves a result for every positive integer n in two steps:

1. **Base case**: show the result is true for $n = 1$.
2. **Inductive step**: assume it is true for $n = k$, then prove it for $n = k + 1$.

If both steps work, the result is true for all n. Often you first make a **conjecture** 猜想 (a sensible guess) from a few cases, then confirm it by **inductive proof** 归纳证明.

Worked example. Prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

Base case $n = 1$: the left side is 1 and the right side is $\frac{1}{2}(1)(2) = 1$. True. **Inductive step**: assume $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$. Then

$$\sum_{r=1}^{k+1} r = \frac{1}{2}k(k+1) + (k+1) = (k+1)\left(\frac{k}{2} + 1\right) = \frac{1}{2}(k+1)(k+2).$$

This is the formula with $n = k + 1$, so by induction it holds for all n.