Further Pure Mathematics 2

A-Level Further Mathematics

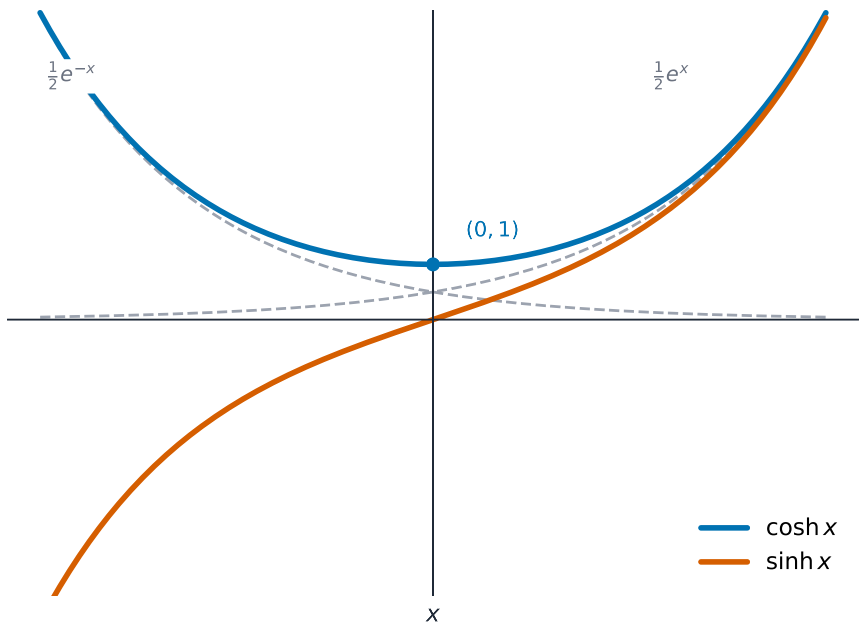
This handout covers Topic 2: **Further Pure Mathematics** 进阶纯数学 2. It adds hyperbolic functions, eigenvalues, new differentiation and integration, de Moivre's theorem, and methods for differential equations.

Hyperbolic functions

A hanging cable forms a catenary —the curve of the hyperbolic cosine.

The **hyperbolic functions** 双曲函数 are built from the exponential function:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x}.$$



cosh *is the average of* e^x *and* e^{-x}*;* sinh *is half their difference.*

They obey identities much like the trigonometric ones, the main one being $\cosh^2 x - \sinh^2 x = 1$. The **inverse hyperbolic functions** 反双曲函数 have a **logarithmic form** 对数形式, for example $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

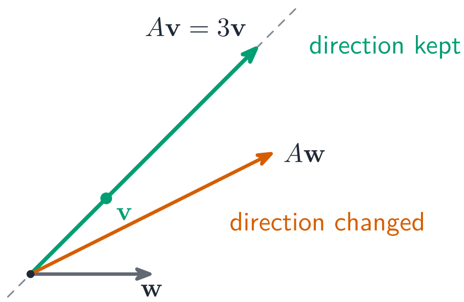
Worked example. Show that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1.$$

Matrices

You can write three linear equations in three unknowns as a single **matrix equation** 矩阵方程 $A\mathbf{x} = \mathbf{b}$. If A is non-singular there is one solution; if A is singular the equations are either inconsistent (no solution) or have infinitely many.

For a square matrix A, the **characteristic equation** 特征方程 is $\det(A - \lambda I) = 0$. Its solutions are the **eigenvalues** 特征值 λ, and for each one the vector $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$ is an **eigenvector** 特征向量. You can then write $A = QDQ^{-1}$, where D is a **diagonal matrix** 对角矩阵 of eigenvalues and the columns of Q are the eigenvectors.



Multiplying an eigenvector by A only stretches it; a general vector is turned to a new direction.

Worked example. Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = 0 \;\Rightarrow\; 2-\lambda = \pm 1.$$

So $\lambda = 1$ or $\lambda = 3$. (The eigenvector for $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $\lambda = 1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

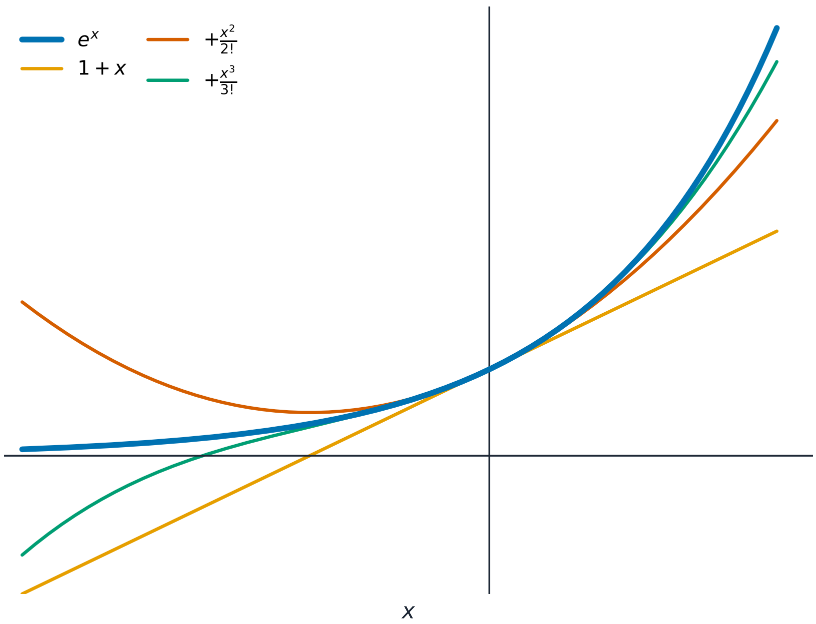
Differentiation

You can now differentiate the hyperbolic functions and the inverse functions $\sin^{-1} x$, $\tan^{-1} x$, $\sinh^{-1} x$ and so on. You can also find $\dfrac{d^2y}{dx^2}$ for curves given implicitly or parametrically.

A **Maclaurin's series** 麦克劳林级数 writes a function as a power series:

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

For example $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$



Adding more terms makes the polynomial match e^x over a wider stretch around $x = 0$.

Integration

Learn these standard integrals:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C.$$

A trigonometric or hyperbolic substitution handles related forms. A **reduction formula** 递推公式 links an integral I_n to I_{n-1}, so you can work down step by step. Integration also gives the **arc length** 弧长 of a curve and the **surface area of revolution** 旋转曲面面积 when a curve is turned about an axis.

Worked example. Find $\int \frac{1}{\sqrt{4-x^2}}\,dx$.

Here $a^2 = 4$, so $a = 2$ and

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + C.$$

Complex numbers

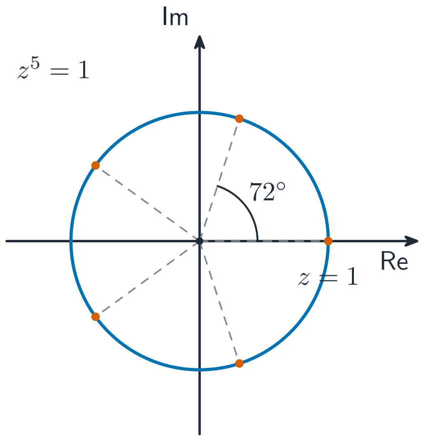
Self-similar patterns like Romanesco broccoli arise from iterating functions in the complex plane.

Image: Jon Sullivan, Public domain (commons.wikimedia.org)

A **complex number** 复数 in polar form is $z = r(\cos\theta + i\sin\theta)$. **De Moivre's theorem** 棣莫弗定理 says that for any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

It is used to expand $\cos n\theta$ and $\sin n\theta$ in powers, to sum series, and to find the n **roots of unity** 单位根 (the solutions of $z^n = 1$, equally spaced around the unit circle).



The five solutions of $z^5 = 1$ are equally spaced $72°$ apart on the unit circle.

Worked example. Use de Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$.

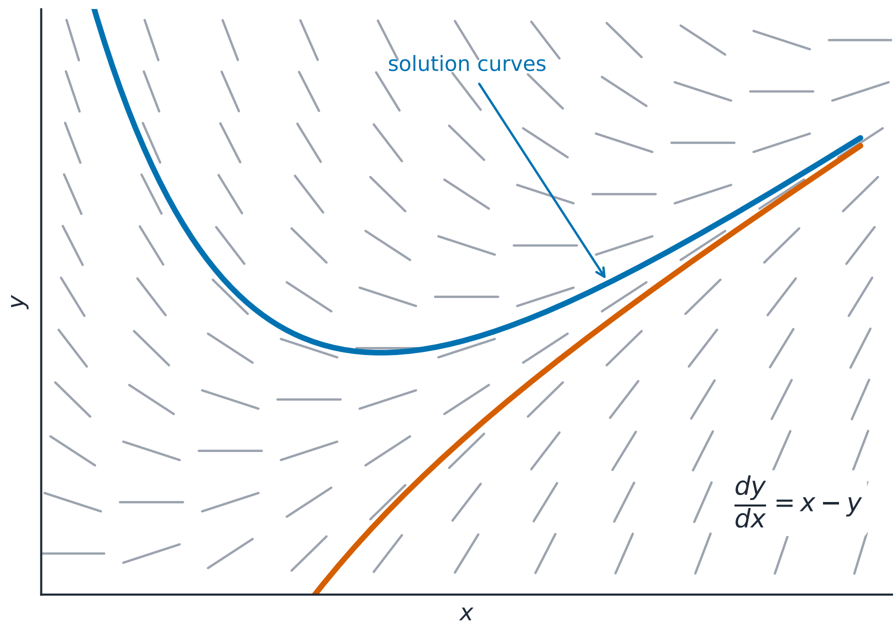
Take the real part of $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta.$$

Differential equations

For a first order linear equation $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, multiply by the **integrating factor** 积分因子 $\mu = e^{\int P\,dx}$. The left side then becomes $\dfrac{d}{dx}(\mu y)$, so you can integrate directly.

For a linear equation with constant coefficients, the **general solution** 通解 is the sum of two parts: the **complementary function** 余函数 (the solution of the equation with the right side set to 0, found from the auxiliary equation) and a **particular integral** 特积分 (any one solution of the full equation).



A solution curve follows the slope field, staying tangent to the little segments everywhere.

Worked example. Solve $\dfrac{dy}{dx} + 2y = e^x$.

The integrating factor is $\mu = e^{\int 2\,dx} = e^{2x}$. Multiplying through gives $\dfrac{d}{dx}(y\,e^{2x}) = e^{3x}$, so

$$y\,e^{2x} = \tfrac{1}{3}e^{3x} + C \;\Rightarrow\; y = \tfrac{1}{3}e^{x} + Ce^{-2x}.$$