Further Mechanics

A-Level Further Mathematics

This handout covers Topic 3: **Further Mechanics** 进阶力学. It extends mechanics to projectiles, rigid bodies, circular motion, elastic strings, variable forces and collisions. Take $g = 10$ m s^{-2}.

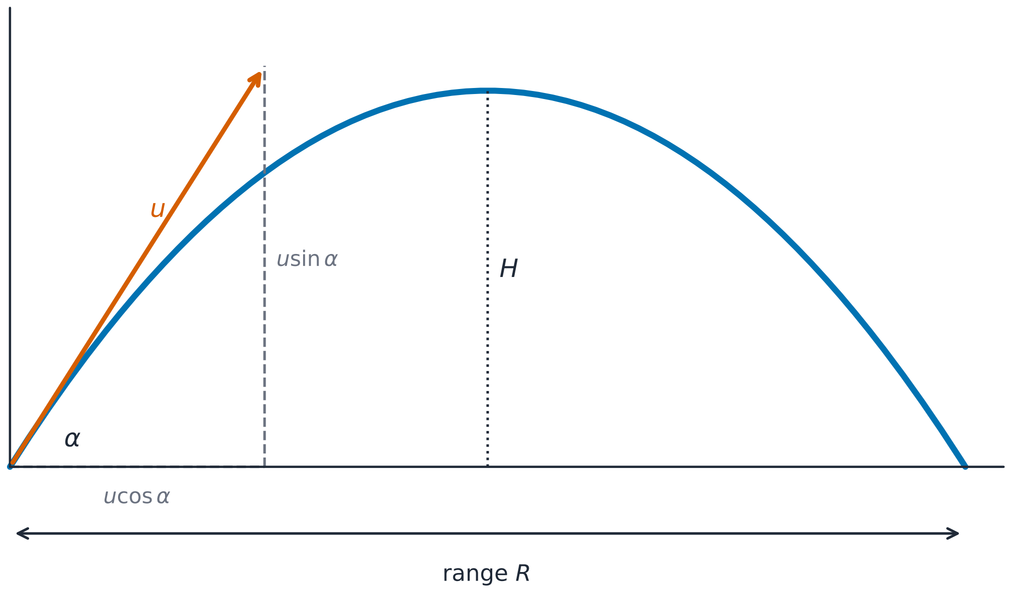
Motion of a projectile

Water jets follow parabolic paths —the classic projectile motion under gravity.

A **projectile** 抛射体 moves freely under gravity, so it has **constant acceleration** 匀加速 g downwards and no horizontal acceleration. Treat the horizontal and vertical motions separately. If it is launched at speed u and angle α:

$$\text{horizontal: } x = u\cos\alpha \cdot t, \qquad \text{vertical: } y = u\sin\alpha \cdot t - \tfrac{1}{2}gt^2.$$

Eliminating t gives the **Cartesian equation** 直角坐标方程 of the **trajectory** 轨迹 (the path), which is a parabola. The range on level ground is $\dfrac{u^2\sin 2\alpha}{g}$ and the greatest height is $\dfrac{u^2\sin^2\alpha}{2g}$.



The path is a parabola; the launch speed u splits into a steady horizontal part and a vertical part slowed by gravity.

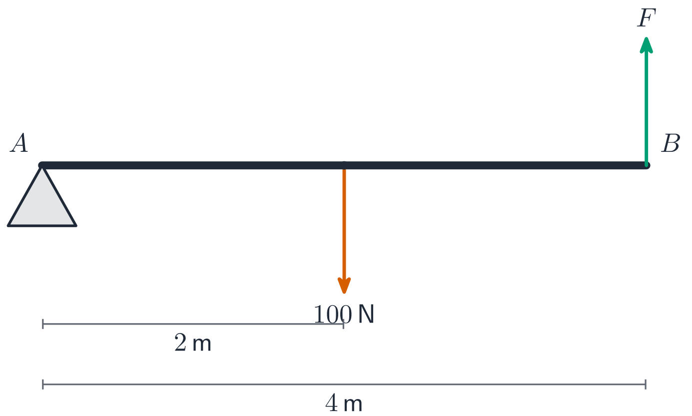
Worked example. A ball is thrown at $u = 20$ m s^{-1} at 30° to the horizontal. Find the range and greatest height.

$$\text{range} = \frac{20^2 \sin 60^\circ}{10} = 40 \times 0.866 = 34.6\text{ m}, \qquad \text{height} = \frac{20^2 \sin^2 30^\circ}{2 \times 10} = \frac{400 \times 0.25}{20} = 5\text{ m}.$$

Equilibrium of a rigid body

The **moment** 力矩 of a force about a point is force × perpendicular distance; it measures turning effect. The weight of a body acts at its **centre of mass** 质心, which you can find using **symmetry** 对称 for a uniform shape, or by treating a composite body as a set of particles.

A rigid body is in **equilibrium** 平衡 when two conditions both hold: the vector sum of the forces is zero, and the sum of the moments about any point is zero. A body may also be on the edge of **toppling** 翻倒 (turning over) or **sliding** 滑动 (slipping).



Taking moments about the pivot A balances the turning effects: $F \times 4 = 100 \times 2$.

Worked example. A uniform beam AB of length 4 m and weight 100 N rests on a pivot at A. A vertical force F at B keeps it horizontal. Find F.

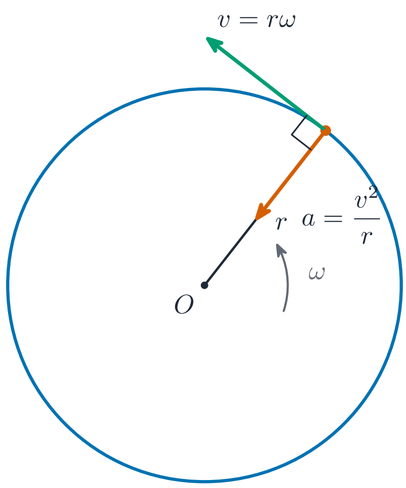
Take moments about A (the weight acts at the centre, 2 m from A):

$$F \times 4 = 100 \times 2 \;\Rightarrow\; F = 50 \text{ N}.$$

Circular motion

For a particle moving in a circle of radius r, the **angular speed** 角速度 ω links to the speed by $v = r\omega$. The acceleration points towards the centre —the **centripetal acceleration** 向心加速度—with size

$$a = r\omega^2 = \frac{v^2}{r}.$$



The velocity points along the tangent; the acceleration points inward to the centre.

In a **horizontal circle** 水平圆 the speed is constant. In a **vertical circle** 竖直圆 use energy conservation, because the speed changes with height.

Worked example. A particle moves in a horizontal circle of radius 2 m with angular speed 3 rad s^{-1}. Find its speed and acceleration.

$$v = r\omega = 2 \times 3 = 6 \text{ m s}^{-1}, \qquad a = r\omega^2 = 2 \times 3^2 = 18 \text{ m s}^{-2}.$$

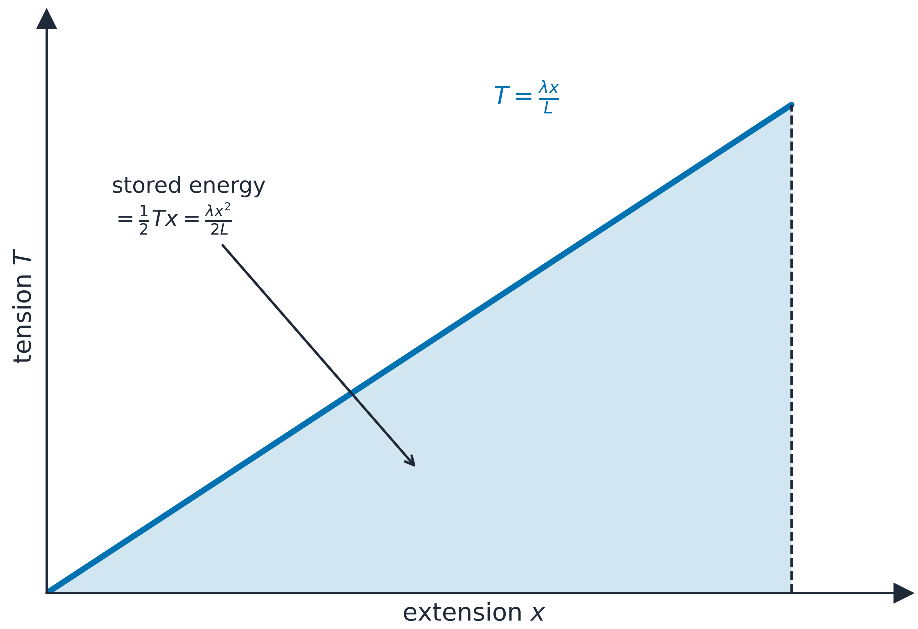
Hooke's law

Hooke's law 胡克定律 says the tension in an elastic string or spring is proportional to its extension x:

$$T = \frac{\lambda x}{L},$$

where L is the natural length and λ is the **modulus of elasticity** 弹性模量. Stretching stores **elastic potential energy** 弹性势能:

$$E = \frac{\lambda x^2}{2L}.$$



Tension rises in proportion to extension; the shaded triangle is the stored elastic energy.

Worked example. An elastic string of natural length 2 m and modulus 50 N is stretched by 0.5 m. Find the tension and the stored energy.

$$T = \frac{50 \times 0.5}{2} = 12.5 \text{ N}, \qquad E = \frac{50 \times 0.5^2}{2 \times 2} = 3.125 \text{ J}.$$

Linear motion under a variable force

When the force depends on position x, use acceleration in the form $a = v\dfrac{dv}{dx}$, which turns Newton's law into a **differential equation 微分方程** relating v and x.

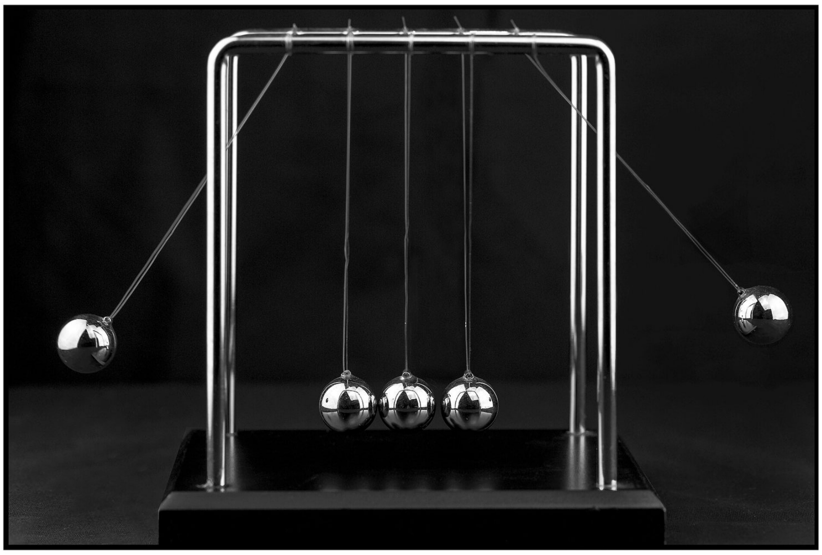
Worked example. A particle of mass 8 kg moves along a line under a **variable force 变力** of size $(x^3 + 4x)$ N acting in the direction of motion. When $x = 0$, $v = 1$. Find v in terms of x.

Newton's law gives $8v\dfrac{dv}{dx} = x^3 + 4x$. Separating and integrating:

$$\int 8v\,dv = \int (x^3 + 4x)\,dx \;\Rightarrow\; 4v^2 = \tfrac{1}{4}x^4 + 2x^2 + C.$$

At $x = 0$, $v = 1$ gives $C = 4$, so $4v^2 = \frac{1}{4}x^4 + 2x^2 + 4 = 4\left(\frac{x^2}{4} + 1\right)^2$. Hence $v = \frac{1}{4}x^2 + 1$.

Momentum

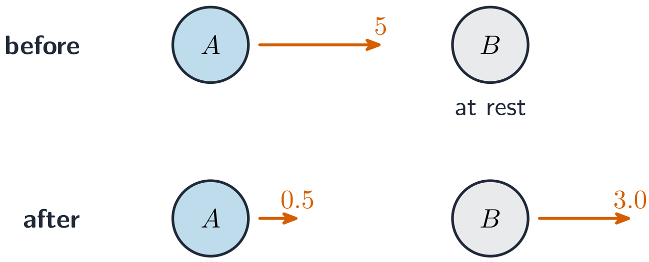


A Newton's cradle demonstrates conservation of momentum in collisions.

In a collision, total momentum is conserved —the **conservation of momentum 动量守恒**. The bounciness is measured by **Newton's experimental law 牛顿实验定律**, which defines the **coefficient of restitution 恢复系数** e:

$$e = \frac{\text{speed of separation}}{\text{speed of approach}}, \qquad 0 \leqslant e \leqslant 1.$$

Here $e = 1$ is perfectly elastic (no energy lost) and $e = 0$ is inelastic (the bodies stay together).



$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{3.0 - 0.5}{5} = 0.5$$

The coefficient of restitution e compares how fast the bodies separate with how fast they approached.

Worked example. A sphere A of mass 2 kg moving at 5 m s^{-1} hits a stationary sphere B of mass 3 kg, with $e = 0.5$. Find the speeds afterwards.

Momentum: $2(5) = 2v_A + 3v_B$, so $2v_A + 3v_B = 10$. Restitution: $v_B - v_A = 0.5(5) = 2.5$. Solving together gives $v_A = 0.5$ m s^{-1} and $v_B = 3.0$ m s^{-1}.