Pure Mathematics 1

A-Level Mathematics

This handout covers Topic 1: **Pure Mathematics** 纯数学 1. It is the algebra 代数 and calculus 微积分 core of the course. Each ## section is one syllabus subtopic.

Quadratics

A suspension bridge: the main cable hangs in a parabola.

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A **quadratic** 二次式 is an expression of the form ax^2+bx+c, where $a \neq 0$. The letters a, b, c are the **coefficients** 系数 (the fixed numbers). Much of this section is about solving the **equation** $ax^2+bx+c=0$.

Completing the square

To **complete the square** 配方 means to write the quadratic in the form

$$a(x+p)^2+q.$$

This form is useful: it shows the **vertex** 顶点 (turning point) of the curve at $(-p,\ q)$, and it gives a quick way to solve the equation.

Worked example. Write $9x^2-36x+8$ in the form $p(x+q)^2+r$.

Take the factor 9 out of the first two terms, then complete the square inside:

$$\begin{aligned} 9x^2-36x+8 &= 9\left(x^2-4x\right)+8 \\ &= 9\left((x-2)^2-4\right)+8 \\ &= 9(x-2)^2-36+8 = 9(x-2)^2-28. \end{aligned}$$

So $p=9$, $q=-2$, $r=-28$.

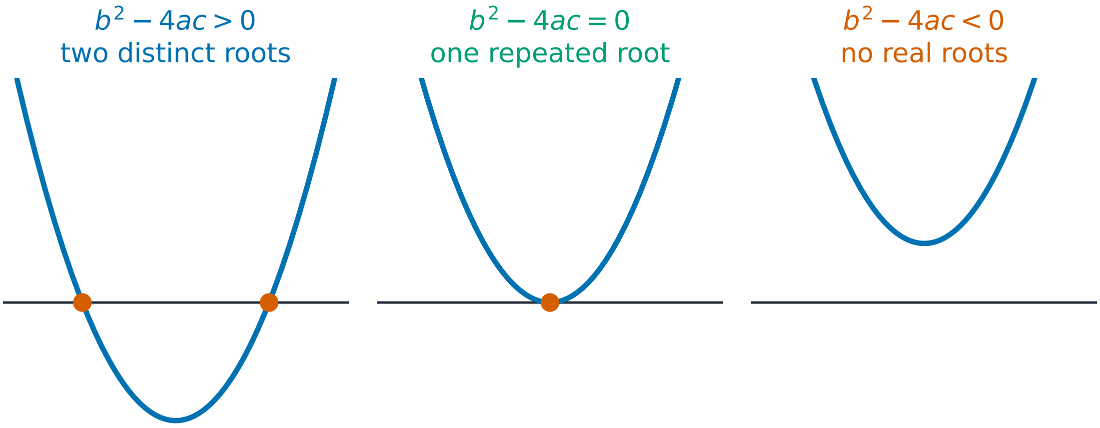
The discriminant

The **discriminant** 判别式 of ax^2+bx+c is

$$\Delta = b^2-4ac.$$

It tells you how many **real roots** 实根 (real solutions) the equation $ax^2 + bx + c = 0$ has:

Discriminant	**Roots**
$b^2 - 4ac > 0$	two **distinct** 相异 real roots
$b^2 - 4ac = 0$	one repeated real root
$b^2 - 4ac < 0$	no real roots



The sign of $b^2 - 4ac$ decides how many times the parabola meets the x-axis.

Worked example. Find the values of the constant k for which $3kx^2 + (k+8)x + 3 = 0$ has two distinct real roots.

Here $a = 3k$, $b = k + 8$, $c = 3$. For two distinct real roots you need $b^2 - 4ac > 0$:

$$(k+8)^2 - 4(3k)(3) > 0 \;\Rightarrow\; k^2 + 16k + 64 - 36k > 0 \;\Rightarrow\; k^2 - 20k + 64 > 0.$$

Factorise: $(k-4)(k-16) > 0$, so $k < 4$ or $k > 16$. You also need $a \neq 0$, so $k \neq 0$.

Quadratic equations and inequalities

To solve a **quadratic equation** 二次方程, factorise, complete the square, or use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To solve a **quadratic inequality** 二次不等式 such as $(k-4)(k-16) > 0$, find the two roots, then decide which side of each root makes the statement true. A sketch of the parabola 抛物线 helps: the curve is above the x-axis (positive) outside the roots and below it (negative) between them.

Simultaneous equations

To solve a pair of **simultaneous equations** 联立方程 where one is linear and one is quadratic, use **substitution** 代入: rearrange the linear equation for one letter, then put that into the quadratic. This gives a single quadratic to solve.

Equations that are quadratic in disguise

Some equations are quadratic in **some function of** x. For example $x^4 - 5x^2 + 4 = 0$ is quadratic in x^2: let $u = x^2$, solve $u^2 - 5u + 4 = 0$, then go back to x. You will use this idea again in trigonometry.

Functions

A **function** 函数 is a rule that sends each input to exactly one output. Write it as $f(x)$.

- The **domain** 定义域 is the set of allowed inputs x.
- The **range** 值域 is the set of outputs the function actually produces.

A function is **one-one** 一一对应 if different inputs always give different outputs. (No output is repeated.) Only a one-one function has an **inverse function** 反函数 f^{-1}, which reverses the rule.

The **composition** 复合 of two functions means doing one after the other. $fg(x)$ means "do g first, then f": $fg(x) = f(g(x))$. The composite function fg exists only when the range of g lies inside the domain of f.

Finding an inverse

To find f^{-1}: write $y = f(x)$, make x the subject, then swap letters.

Worked example. The function $f(x) = (x+3)^2 - 12$ is defined for $x \geqslant 0$. Find $f^{-1}(x)$.

Write $y = (x+3)^2 - 12$ and solve for x:

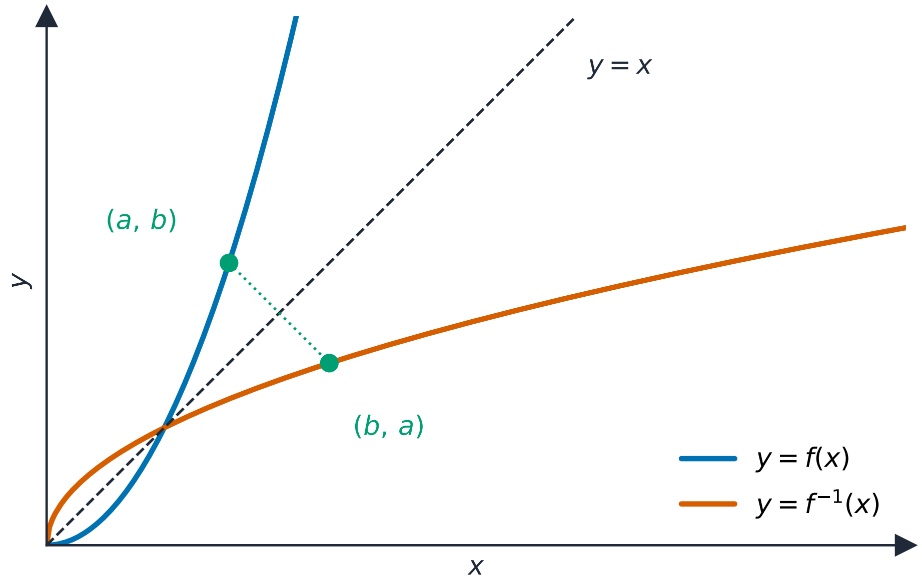
$$(x+3)^2 = y + 12 \;\Rightarrow\; x + 3 = \sqrt{y+12} \;\Rightarrow\; x = \sqrt{y+12} - 3.$$

You take the **positive** square root because $x \geqslant 0$ means $x + 3 \geqslant 3 > 0$. So

$$f^{-1}(x) = \sqrt{x+12} - 3.$$

Graphs of inverses and transformations

The graph of $y = f^{-1}(x)$ is the **reflection** 反射 of $y = f(x)$ in the line $y = x$.

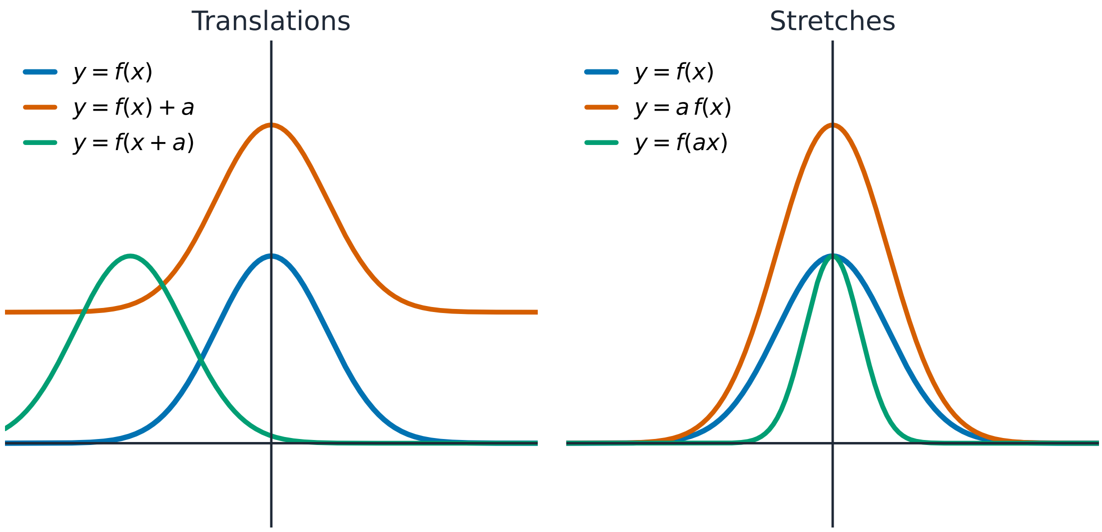


Reflecting $y = f(x)$ in the line $y = x$ gives its inverse; a point (a, b) becomes (b, a).

You should know these **transformations** 变换 of $y = f(x)$:

New equation	Effect on the graph
$y = f(x) + a$	**translation** 平移 up by a
$y = f(x + a)$	translation left by a
$y = a\,f(x)$	**stretch** 伸缩 in the y-direction, scale factor a
$y = f(ax)$	stretch in the x-direction, scale factor $\frac{1}{a}$

When two transformations are combined, the order can matter. State each one fully (type, direction, and amount).



Adding to the output or input slides the curve; a multiplier stretches it.

Coordinate geometry

Coordinate geometry 坐标几何 studies lines and circles using their equations.

Straight lines

The **gradient** 斜率 (steepness) of the line joining (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

You can write the **equation of a straight line** 直线方程 in any of these forms:

$$y = mx + c, \qquad y - y_1 = m(x - x_1), \qquad ax + by + c = 0.$$

Two lines are **parallel** 平行 when their gradients are equal, and **perpendicular** 垂直 (at right angles) when the product of their gradients is -1.

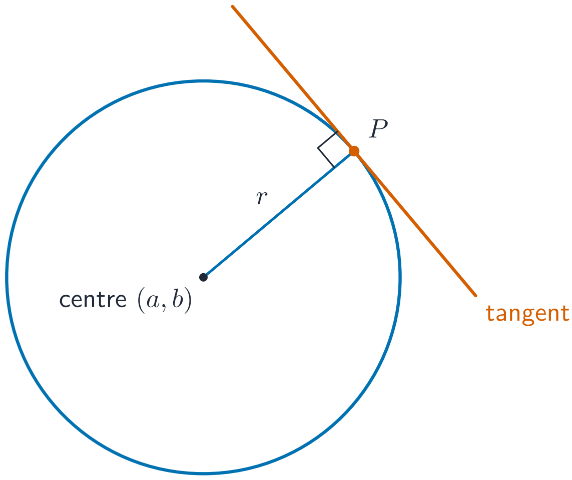
Circles

The **circle** 圆 with **centre** 圆心 (a, b) and **radius** 半径 r has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

An expanded form like $x^2 + y^2 - 6x + 10y - 27 = 0$ is the same circle: complete the square in x and in y to find the centre and radius.

A **tangent** 切线 to a circle touches it at one point and is perpendicular to the radius at that point. This right-angle fact solves most circle problems.



A tangent touches the circle once and meets the radius at a right angle.

Worked example. The points $P(1, 1)$ and $Q(7, 11)$ are the ends of a **diameter** 直径 of a circle. Find the equation of the circle.

The centre is the midpoint of PQ:

$$\left(\frac{1+7}{2}, \frac{1+11}{2} \right) = (4, 6).$$

The radius is half the length of PQ:

$$r = \tfrac{1}{2}\sqrt{(7-1)^2 + (11-1)^2} = \tfrac{1}{2}\sqrt{36 + 100} = \tfrac{1}{2}\sqrt{136} = \sqrt{34}.$$

So the circle is $(x - 4)^2 + (y - 6)^2 = 34$.

Circular measure

Radians

A **radian** 弧度 is another way to measure angles. One radian is the angle at the centre of a circle that cuts off an **arc** 弧 equal in length to the radius. The link between radians and **degrees** 度 is

$$\pi \text{ radians} = 180^\circ.$$

So to change degrees to radians, multiply by $\dfrac{\pi}{180}$; to change radians to degrees, multiply by $\dfrac{180}{\pi}$.

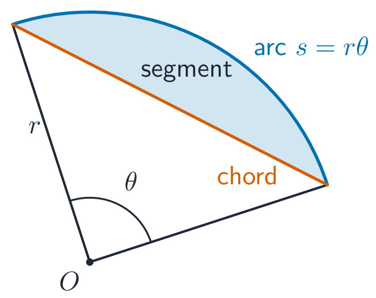
Arc length and sector area

For a **sector** 扇形 with radius r and angle θ **in radians**:

$$\text{arc length} = s = r\theta, \qquad \text{sector area} = A = \tfrac{1}{2}r^2\theta.$$

A **chord** 弦 cuts the sector into a triangle and a **segment** 弓形. The segment area is the sector minus the triangle:

$$\text{segment} = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta = \tfrac{1}{2}r^2(\theta - \sin\theta).$$



The shaded segment is the part of the sector between the chord and the arc.

Worked example. A sector has centre O and the angle at O is $\frac{2}{3}\pi$ radians. Show that the segment cut off by the chord has area about $0.614r^2$.

$$\text{segment} = \tfrac{1}{2}r^2\left(\tfrac{2}{3}\pi - \sin\tfrac{2}{3}\pi\right) = \tfrac{1}{2}r^2(2.0944 - 0.8660) = \tfrac{1}{2}r^2(1.2284) \approx 0.614r^2.$$

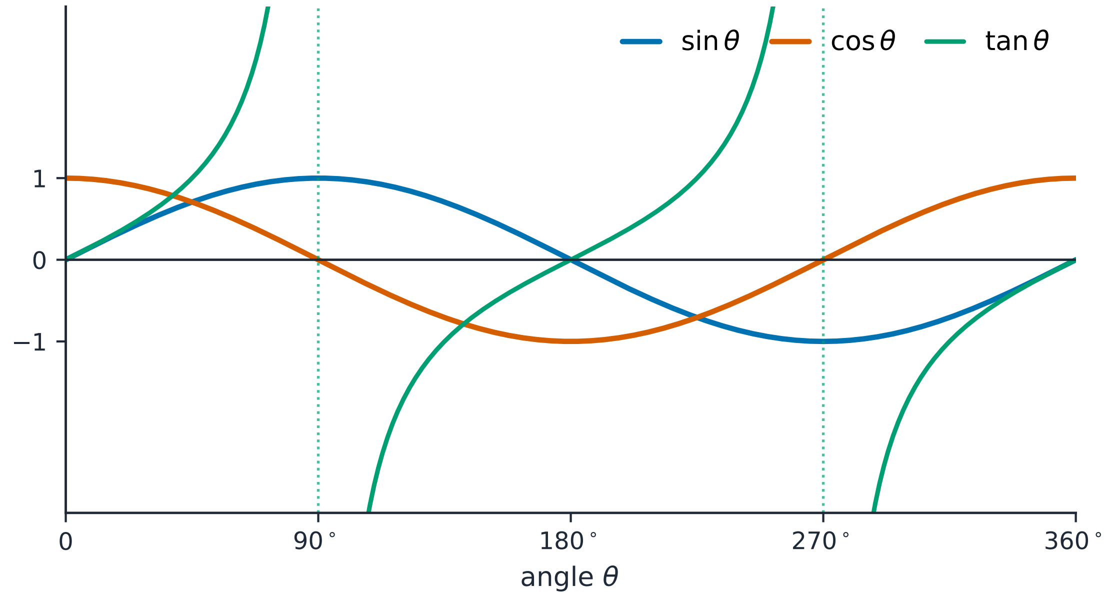
Trigonometry

A Ferris wheel: a point on the rim rises and falls like a sine curve.

Graphs and exact values

You must know the shape of the graphs of the **sine** 正弦, **cosine** 余弦 and **tangent function** 正切 (written sin, cos, tan). The sine and cosine graphs wave between -1 and 1 and repeat every 360° (2π). Learn these exact values:

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$



Over one turn sin *and* cos *stay between* -1 *and* 1*;* tan *shoots off at* $90°$ *and* $270°$*.*

The notations $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ mean the **inverse** angle (the **principal value** 主值).

Identities

An **identity** 恒等式 is true for every value of the angle. The two you must know are

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}, \qquad \sin^2\theta + \cos^2\theta \equiv 1.$$

Use them to rewrite an equation so it contains only one trig function.

Solving trigonometric equations

To solve a **trigonometric equation** 三角方程, first reduce it to one function, then find every solution in the given interval.

Worked example. Solve $6\sin\theta = 1 + \dfrac{2}{\sin\theta}$ for $-180° < \theta < 180°$.

Multiply through by $\sin\theta$ to clear the fraction. This makes a **quadratic in** $\sin\theta$:

$$6\sin^2\theta - \sin\theta - 2 = 0 \;\Rightarrow\; (3\sin\theta - 2)(2\sin\theta + 1) = 0.$$

So $\sin\theta = \frac{2}{3}$ or $\sin\theta = -\frac{1}{2}$.

- $\sin\theta = \frac{2}{3}$: $\theta = 41.8°$ or $\theta = 180° - 41.8° = 138.2°$.
- $\sin\theta = -\frac{1}{2}$: $\theta = -30°$ or $\theta = -150°$.

The four solutions are $\theta = -150°,\ -30°,\ 41.8°,\ 138.2°$.

Series

The binomial expansion

For a positive integer n, the **binomial expansion 二项展开式** is

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n,$$

where $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$ is a **binomial coefficient 二项式系数**.

Worked example. Find the first three terms, in ascending powers of x, of $(2-px)^5$.

$$(2-px)^5 = 2^5 + \binom{5}{1}2^4(-px) + \binom{5}{2}2^3(-px)^2 + \cdots = 32 - 80px + 80p^2x^2 + \cdots$$

Arithmetic and geometric progressions

A **progression 数列** (sequence) is a list of terms following a rule.

- An **arithmetic progression 等差数列** (AP) adds a fixed **common difference 公差** d each step. The nth **term 项** is $u_n = a + (n-1)d$, and the sum of the first n terms is $S_n = \frac{n}{2}\big(2a + (n-1)d\big)$.
- A **geometric progression 等比数列** (GP) multiplies by a fixed **common ratio 公比** r each step. The nth term is $u_n = ar^{n-1}$, and $S_n = \dfrac{a(1-r^n)}{1-r}$.

A GP **converges 收敛** (settles to a limit) when $|r| < 1$. Then it has a **sum to infinity 无穷和**

$$S_\infty = \frac{a}{1-r}.$$

Worked example. The third term of a GP is 18 and the sum of the first three terms is 26. The common ratio is negative. Find the sum to infinity.

From $ar^2 = 18$ you get $a = \dfrac{18}{r^2}$. Put this into $a(1+r+r^2) = 26$:

$$18(1+r+r^2) = 26r^2 \;\Rightarrow\; 8r^2 - 18r - 18 = 0 \;\Rightarrow\; (4r+3)(r-3) = 0.$$

The ratio is negative, so $r = -\frac{3}{4}$ and $a = \dfrac{18}{(3/4)^2} = 32$. Then

$$S_\infty = \frac{32}{1-(-\frac{3}{4})} = \frac{32}{\frac{7}{4}} = \frac{128}{7}.$$

Differentiation

Differentiation 微分 finds the **gradient of a curve 曲线斜率** at each point. The gradient is the **limit 极限** of the gradients of shorter and shorter **chords**, called the **derivative 导数**.

The rules

Write the derivative as $f'(x)$ or $\dfrac{dy}{dx}$. The basic rule is

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \quad \text{for any rational } n.$$

Differentiate sums term by term, and use the **chain rule** 链式法则 for a function inside a function:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

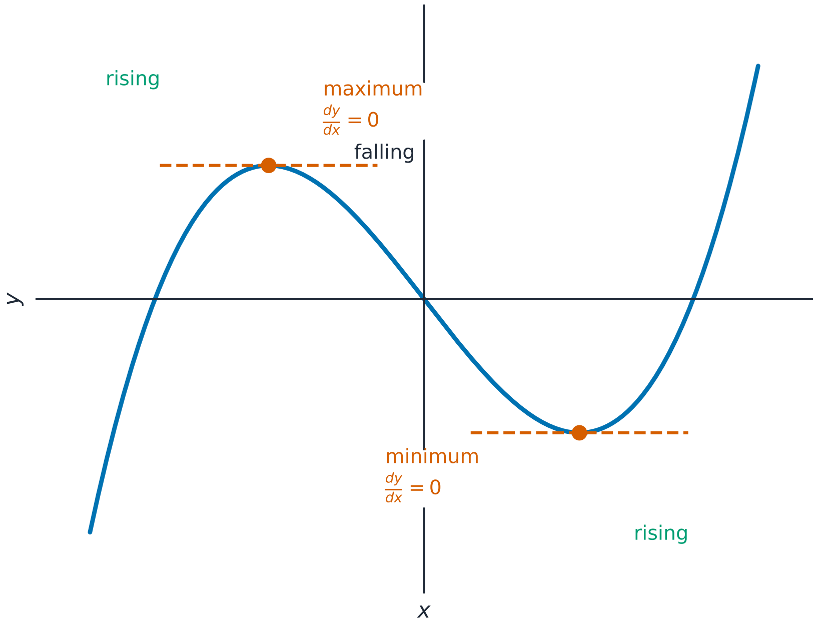
Differentiating again gives the **second derivative** 二阶导数 $f''(x)$ or $\dfrac{d^2y}{dx^2}$.

Using the derivative

- **Tangent and normal.** The gradient of the curve at a point is the gradient of the **tangent** there. The **normal** 法线 is perpendicular to the tangent, so its gradient is $-\dfrac{1}{(\text{tangent gradient})}$.
- **Increasing or decreasing.** The function is an **increasing function** 增函数 where $\dfrac{dy}{dx} > 0$, and a **decreasing function** 减函数 where $\dfrac{dy}{dx} < 0$.
- **Rate of change.** A derivative is a **rate of change** 变化率. Linked rates use the chain rule, e.g. $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$.

Stationary points

A **stationary point** 驻点 is where $\dfrac{dy}{dx} = 0$. Test its nature with the second derivative: $f''(x) > 0$ gives a **minimum point** 极小值点, and $f''(x) < 0$ gives a **maximum point** 极大值点.



At a maximum or a minimum the tangent is flat, so $\frac{dy}{dx} = 0$.

Worked example. The curve $y = 4x^{1/2} - x$ has a maximum point at $x = a$. Find a.

$$\frac{dy}{dx} = 2x^{-1/2} - 1 = 0 \;\Rightarrow\; \frac{2}{\sqrt{x}} = 1 \;\Rightarrow\; \sqrt{x} = 2 \;\Rightarrow\; x = 4.$$

So $a = 4$.

Integration

Integration 积分 is the reverse of differentiation. Reversing the power rule gives

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad (n \neq -1).$$

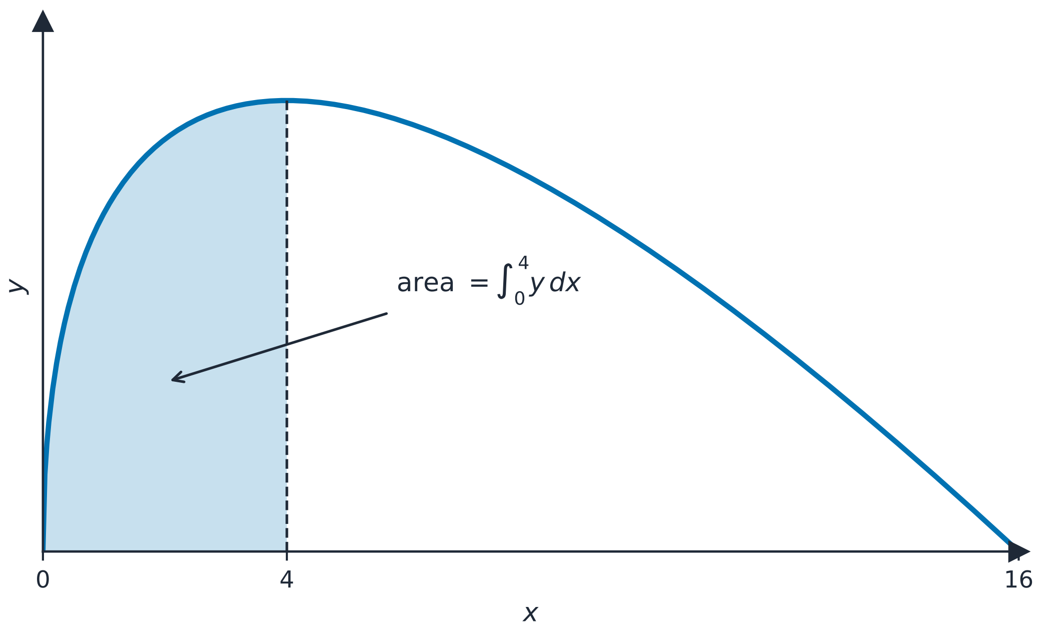
The $+C$ is the **constant of integration** 积分常数. If you know one point on the curve, substitute it to find C.

Definite integrals and area

A **definite integral** 定积分 has limits and gives a number:

$$\int_p^q f(x)\,dx = \big[F(x)\big]_p^q = F(q) - F(p).$$

The area of the **region** 区域 between a curve and the x-axis, from $x = p$ to $x = q$, is $\int_p^q y\,dx$. For the area between two curves, integrate (top curve − bottom curve).



The definite integral $\int_0^4 y\,dx$ is the shaded area under the curve.

Worked example. The curve $y = 4x^{1/2} - x$ meets the x-axis again at $x = 16$. Find the area between the curve and the x-axis from $x = 0$ to $x = 4$.

$$\int_0^4 \left(4x^{1/2} - x\right) dx = \left[\frac{8}{3}x^{3/2} - \frac{x^2}{2}\right]_0^4 = \frac{8}{3}(8) - \frac{16}{2} = \frac{64}{3} - 8 = \frac{40}{3}.$$

Volume of revolution

When a region is turned all the way around an axis it sweeps out a solid. The **volume of revolution** 旋转体体积 about the x-axis is

$$V = \pi \int_p^q y^2 \, dx,$$

and about the y-axis it is $V = \pi \displaystyle\int x^2 \, dy$. For example, the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$, turned about the x-axis, has volume $\pi \displaystyle\int_0^4 x \, dx = \pi \left[\tfrac{x^2}{2} \right]_0^4 = 8\pi$.