Pure Mathematics 3

A-Level Mathematics

This handout covers Topic 3: **Pure Mathematics** 纯数学 3. Subtopics 3.1–3.6 build on the algebra, logarithms, trigonometry, differentiation, integration and numerical methods of Pure Mathematics 2, so this handout explains what is **new** in Pure 3 and then covers the three big new areas: vectors, differential equations and complex numbers.

Algebra

Two new tools join the algebra from Pure 2.

Partial fractions

A single fraction with a factorised bottom can be split into a sum of simpler fractions. This is called writing it in **partial fractions** 部分分式, and it makes a **rational function** 有理函数 (a fraction of polynomials) easy to integrate or expand. Match the form to the bottom:

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b}+\frac{B}{cx+d}, \qquad \frac{1}{(ax+b)(cx+d)^2} = \frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}.$$

Worked example. Express $\dfrac{x+4}{(x+1)(x-2)}$ in partial fractions.

Write $\dfrac{x+4}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$, so $x+4 = A(x-2)+B(x+1)$. Put $x=2$: $6=3B$, so $B=2$. Put $x=-1$: $3=-3A$, so $A=-1$. Hence

$$\frac{x+4}{(x+1)(x-2)} = \frac{2}{x-2} - \frac{1}{x+1}.$$

The binomial expansion for a rational power

The **binomial expansion** 二项展开式 also works when the power is a fraction or is negative, as long as $|x|<1$:

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

For example $(1+x)^{1/2} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$ for $|x|<1$.

Logarithms, trigonometry and numerical methods (from Pure 2)

Subtopics 3.2, 3.3 and 3.6 are the same skills you met in Pure 2: the laws of logarithms with e^x and $\ln x$; the identities $\sec^2\theta \equiv 1+\tan^2\theta$ and $\csc^2\theta \equiv 1+\cot^2\theta$, the compound- and double-angle formulae, and the R-form of $a\sin\theta+b\cos\theta$; and solving an equation numerically by a **sign change** 变号 and an **iterative formula** 迭代公式 $x_{n+1}=F(x_n)$. Use them exactly as before.

Differentiation

The methods are those of Pure 2 (the product, quotient and chain rules, with parametric and implicit curves). One derivative is added —the **inverse tangent** 反正切:

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}.$$

Integration

Pure 3 adds several powerful methods.

- A new standard integral: $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$.
- **Partial fractions**: split a rational function first, then integrate each piece as a logarithm.
- The pattern $\dfrac{k\,f'(x)}{f(x)}$: this integrates to $k\ln|f(x)| + C$. For example $\displaystyle\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + C$.
- **Integration by parts** 分部积分, used for a product: $\displaystyle\int u\,\frac{dv}{dx}\,dx = uv - \int v\,\frac{du}{dx}\,dx$.
- **Integration by substitution** 换元积分: a given change of variable turns a hard integral into an easy one.

Worked example. Find $\displaystyle\int x\cos x\,dx$.

Use integration by parts with $u = x$ and $\dfrac{dv}{dx} = \cos x$, so $\dfrac{du}{dx} = 1$ and $v = \sin x$:

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Vectors

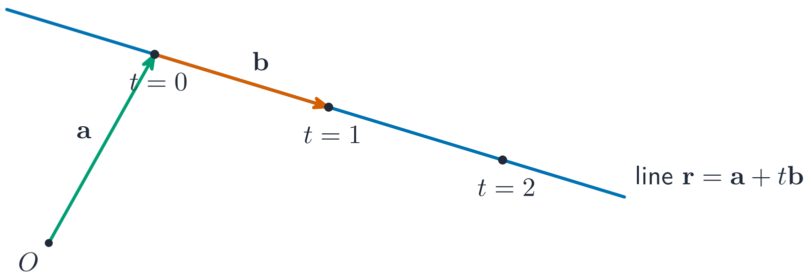
Forces like wind and water are vectors —they have both size and direction.

A **vector** 向量 has both size and direction. Write it as a column, or as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, or as $\overrightarrow{AB}$.

- The **magnitude** 模长 (length) of $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$.
- A **unit vector** 单位向量 has magnitude 1; divide a vector by its magnitude to make one.
- A **position vector** 位置向量 gives a point's place from the origin; a **displacement vector** 位移向量 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ goes from one point to another. Multiplying by a **scalar** 标量 (a plain number) stretches a vector.

Lines and the scalar product

A straight line through point $\mathbf{a}$ in direction $\mathbf{b}$ has vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. Two lines may be **parallel** 平行, may **intersect** 相交 at a point, or may be **skew lines** 异面直线 (not parallel and never meeting).

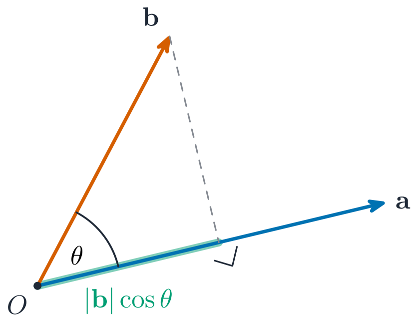


Start at the point $\mathbf{a}$*, then add* t *copies of the direction* $\mathbf{b}$ *to reach any point on the line.*

The **scalar product** 数量积 (dot product) of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ is the angle between them. So $\mathbf{a} \cdot \mathbf{b} = 0$ means the vectors are perpendicular.



The scalar product picks out $|\mathbf{b}| \cos\theta$*, how far* $\mathbf{b}$ *reaches along* $\mathbf{a}$*.*

Worked example. Find the angle between $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

$$\mathbf{a} \cdot \mathbf{b} = (1)(2) + (2)(2) + (2)(1) = 8, \qquad |\mathbf{a}| = |\mathbf{b}| = 3.$$

So $\cos\theta = \dfrac{8}{3 \times 3} = \dfrac{8}{9}$, giving $\theta = 27.3°$.

Differential equations

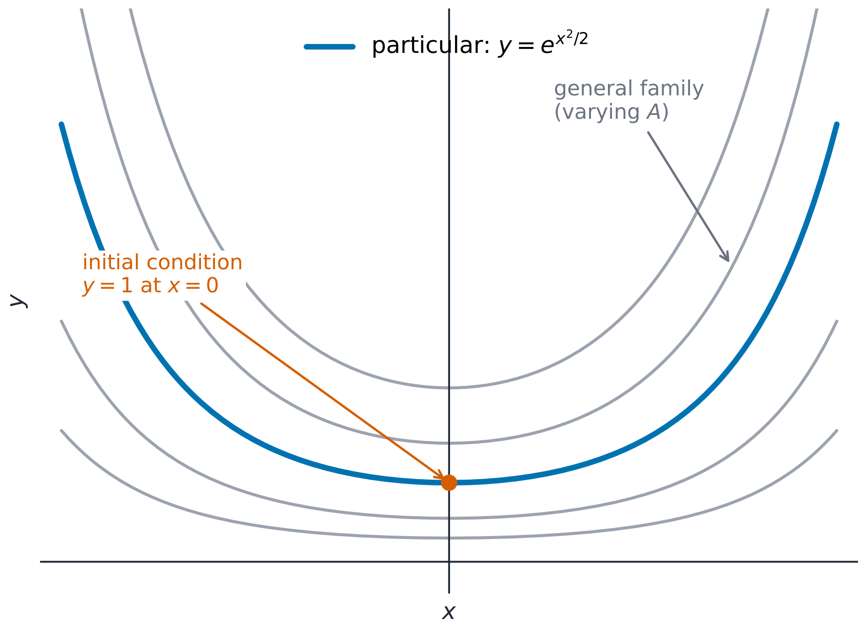
A **differential equation** 微分方程 links a quantity to its **rate of change**. To solve a first-order equation whose variables are **separable** 可分离变量, put all the y terms on one side and all the x terms on the other, then integrate both sides. This gives the **general solution** 通解, which contains a constant. An **initial condition** 初始条件 (a known value) fixes the constant and gives the **particular solution** 特解.

Worked example. Solve $\dfrac{dy}{dx} = xy$, given that $y = 1$ when $x = 0$.

Separate the variables and integrate:

$$\int \frac{1}{y}\,dy = \int x\,dx \;\Rightarrow\; \ln y = \tfrac{1}{2}x^2 + c \;\Rightarrow\; y = Ae^{x^2/2}.$$

Using $y = 1$ at $x = 0$ gives $A = 1$, so $y = e^{x^2/2}$.



The constant A gives a whole family of curves; the condition $y = 1$ at $x = 0$ selects $y = e^{x^2/2}$.

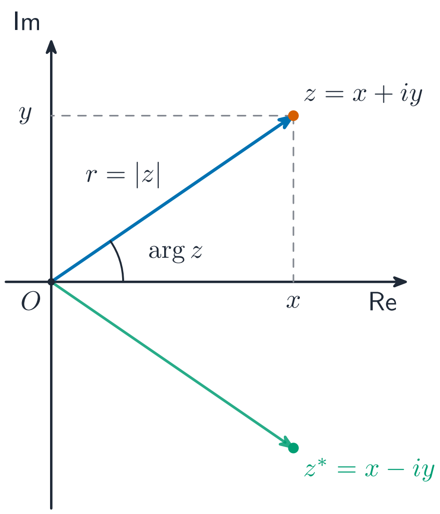
Complex numbers

Self-similar patterns like Romanesco arise from iterating functions in the complex plane.

Image: Jon Sullivan, Public domain (commons.wikimedia.org)

A **complex number** 复数 has the form $z = x + iy$, where $i^2 = -1$. Here x is the **real part** 实部 and y is the **imaginary part** 虚部. This $x + iy$ is the **Cartesian form** 直角坐标形式. Two complex numbers are equal only when their real parts match and their imaginary parts match.

- The **conjugate** 共轭 of $z = x + iy$ is $z^* = x - iy$. For a polynomial with real coefficients, any non-real roots come in conjugate pairs.
- The **modulus** 模 is $|z| = \sqrt{x^2 + y^2}$ (its distance from the origin) and the **argument** 辐角 is the angle the point makes, measured from the positive real axis.
- You can plot z as a point on an **Argand diagram** 阿干图 (the complex plane).
- The **polar form** 极坐标形式 is $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where $r = |z|$ and θ is the argument. Multiplying multiplies the moduli and adds the arguments.



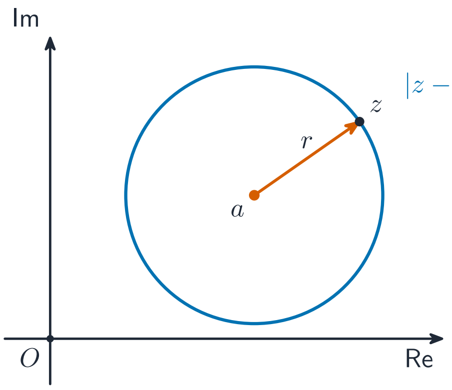
On the Argand diagram $|z|$ is the distance from O, $\arg z$ the angle, and z^* the reflection in the real axis.

To divide, multiply top and bottom by the conjugate of the bottom.

Worked example. Write $\dfrac{3+i}{1-i}$ in the form $x+iy$.

$$\frac{3+i}{1-i} = \frac{(3+i)(1+i)}{(1-i)(1+i)} = \frac{3+3i+i+i^2}{1+1} = \frac{2+4i}{2} = 1+2i.$$

An equation or inequality in z describes a **locus** 轨迹 (a path or region) on the Argand diagram. For example $|z-a|=r$ is a circle of radius r centred at a.



$|z-a|=r$ is the set of points a fixed distance r from a —a circle.