Probability & Statistics 1

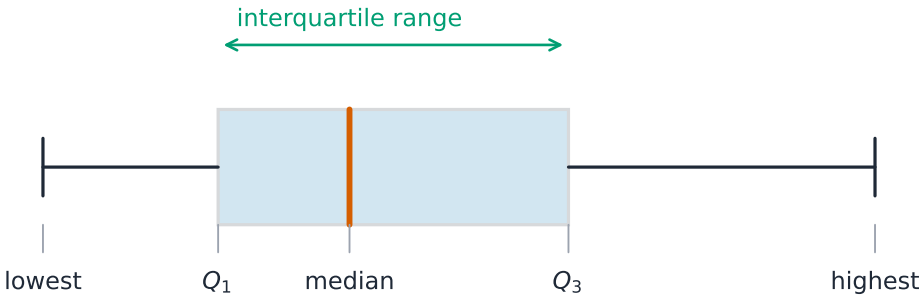
A-Level Mathematics

This handout covers Topic 5: **Probability & Statistics** 概率统计 1. It is about describing data, counting choices, and working out the chance of events.

Representation of data

Choose a diagram that suits the data. You should be able to draw and read:

- a **stem-and-leaf diagram** 茎叶图 (keeps the original values and shows the shape);
- a **box-and-whisker plot** 箱线图 (shows the lowest value, the three quartiles, and the highest value);
- a **histogram** 直方图 (for grouped data, where the **area** of each bar shows the frequency);
- a **cumulative frequency** 累积频数 graph (running totals, used to estimate the median and quartiles).



The box spans the quartiles Q_1 to Q_3; the whiskers reach the lowest and highest values.

Averages and spread

A **measure of central tendency** 集中趋势 is a single "middle" value:

- the **mean** 平均数 $\bar{x} = \dfrac{\sum x}{n}$ (the average);
- the **median** 中位数 (the middle value when the data is in order);
- the **mode** 众数 (the most common value).

A measure of **variation** 离散程度 shows how spread out the data is:

- the **range (of data)** 极差 (highest − lowest);
- the **interquartile range** 四分位距 (upper quartile − lower quartile);
- the **standard deviation** 标准差 $\sigma = \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2}$.

You often work from the totals $\sum x$ and $\sum x^2$. The square of the standard deviation is the **variance** 方差.

Worked example. For 10 values, $\sum x = 50$ and $\sum x^2 = 300$. Find the mean and standard deviation.

$$\bar{x} = \frac{50}{10} = 5, \qquad \sigma = \sqrt{\frac{300}{10} - 5^2} = \sqrt{30 - 25} = \sqrt{5} = 2.24.$$

Permutations and combinations

A **permutation** 排列 is an arrangement where order matters; a **combination** 组合 is a selection where order does not matter. The numbers are

$${}^{n}P_r = \frac{n!}{(n-r)!}, \qquad {}^{n}C_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

To arrange objects in a line when some are repeated, divide by the factorial of each repeat count.

Worked example. How many different arrangements are there of the letters of the word NEEDLESS?

There are 8 letters, with E repeated 3 times and S repeated 2 times:

$$\frac{8!}{3!\,2!} = \frac{40320}{6 \times 2} = 3360.$$

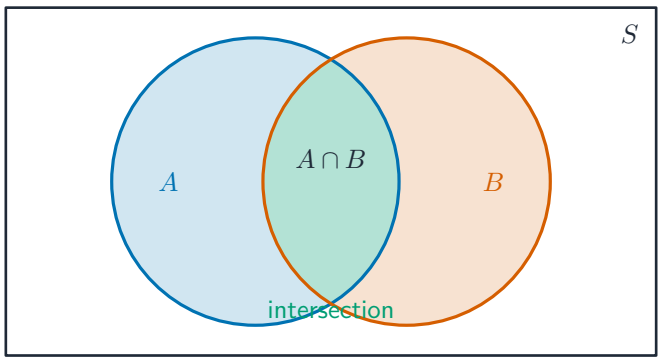
Probability

Dice: a familiar starting point for probability.

Image: Diacritica, CC BY-SA 3.0 (commons.wikimedia.org)

Find a probability by counting equally likely outcomes, or by using permutations and combinations. Combine probabilities with these rules:

- **addition** for "or": $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;
- **multiplication** for "and" when events are independent: $P(A \cap B) = P(A)\,P(B)$.



The overlap of the two circles is $A \cap B$; the addition rule subtracts it once so it is not counted twice.

Two events are **mutually exclusive events** 互斥事件 if they cannot both happen, and **independent events** 独立事件 if one happening does not change the chance of the other. To test independence, check whether $P(A \cap B) = P(A) \times P(B)$. A **conditional probability** 条件概率 is the chance of A given that B has happened: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$.

Worked example. Events have $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cap B) = 0.2$. Are A and B independent?

Test: $P(A) \times P(B) = 0.5 \times 0.4 = 0.2 = P(A \cap B)$. The values are equal, so A and B are independent.

Discrete random variables

A **discrete random variable** 离散型随机变量 X takes separate values, each with a probability. List them in a **probability distribution table** 概率分布表; the probabilities must add to 1. Then the **expectation** 期望 (mean) and **variance** are

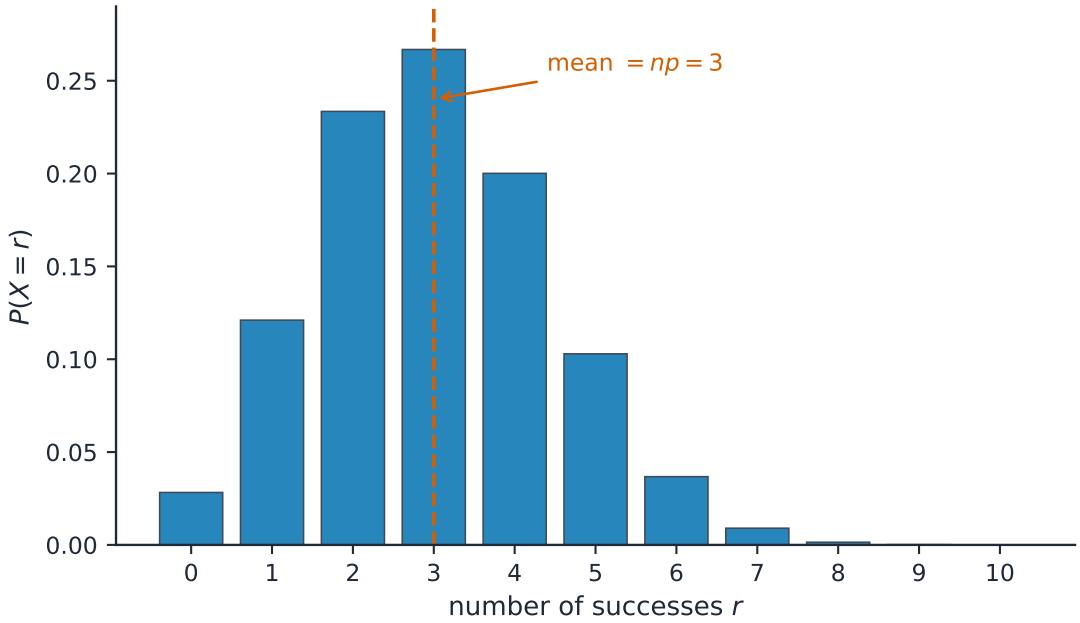
$$E(X) = \sum x\,P(X = x), \qquad \mathrm{Var}(X) = \sum x^2\,P(X = x) - \big(E(X)\big)^2.$$

Two special models:

- the **binomial distribution** 二项分布 $X \sim B(n, p)$, for the number of successes in n independent trials: $P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$, with $E(X) = np$ and $\mathrm{Var}(X) = np(1-p)$;
- the **geometric distribution** 几何分布, for the trial on which the first success happens: $P(X = r) = (1-p)^{r-1}p$, with $E(X) = \dfrac{1}{p}$.

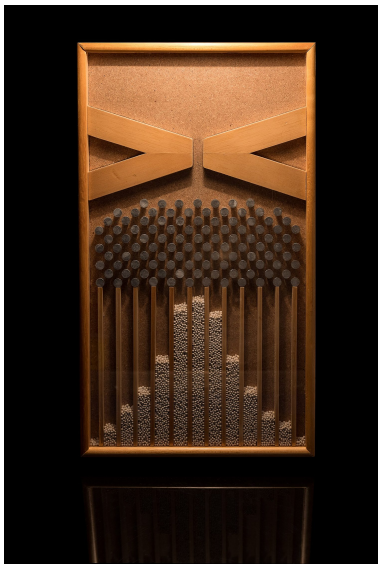
Worked example. $X \sim B(10, 0.3)$. Find $P(X = 2)$ and $E(X)$.

$$P(X = 2) = \binom{10}{2}(0.3)^2(0.7)^8 = 45 \times 0.09 \times 0.05765 = 0.233, \qquad E(X) = 10 \times 0.3 = 3.$$



The distribution $B(10, 0.3)$: each bar is $P(X = r)$, clustered around the mean $np = 3$.

The normal distribution

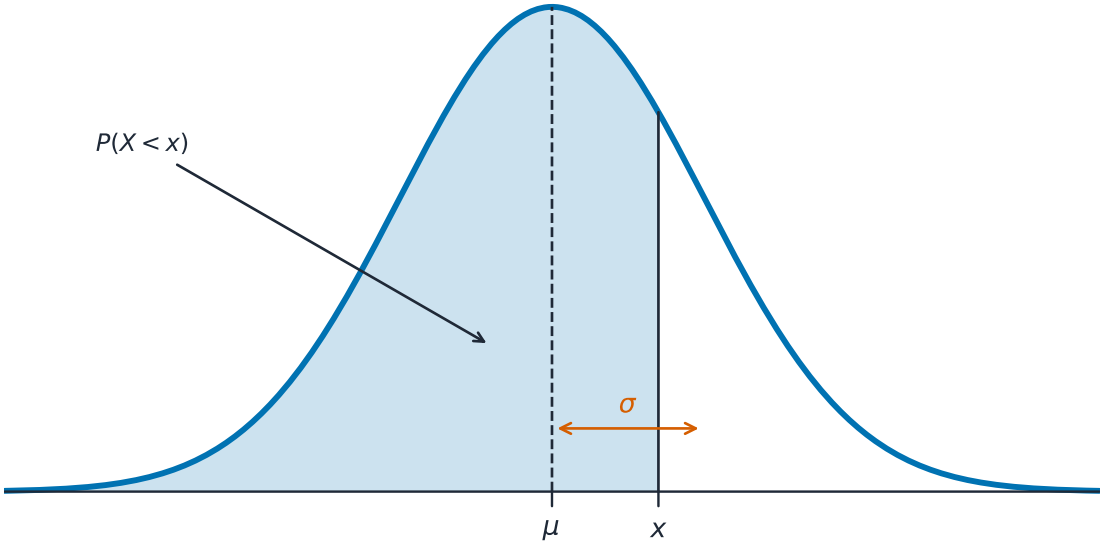


A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.

The **normal distribution** 正态分布 models a **continuous random variable** 连续型随机变量 with a symmetric bell shape. Write $X \sim N(\mu, \sigma^2)$, where μ is the mean and σ is the standard deviation. To use the tables, **standardize** 标准化 to the variable $Z \sim N(0, 1)$:

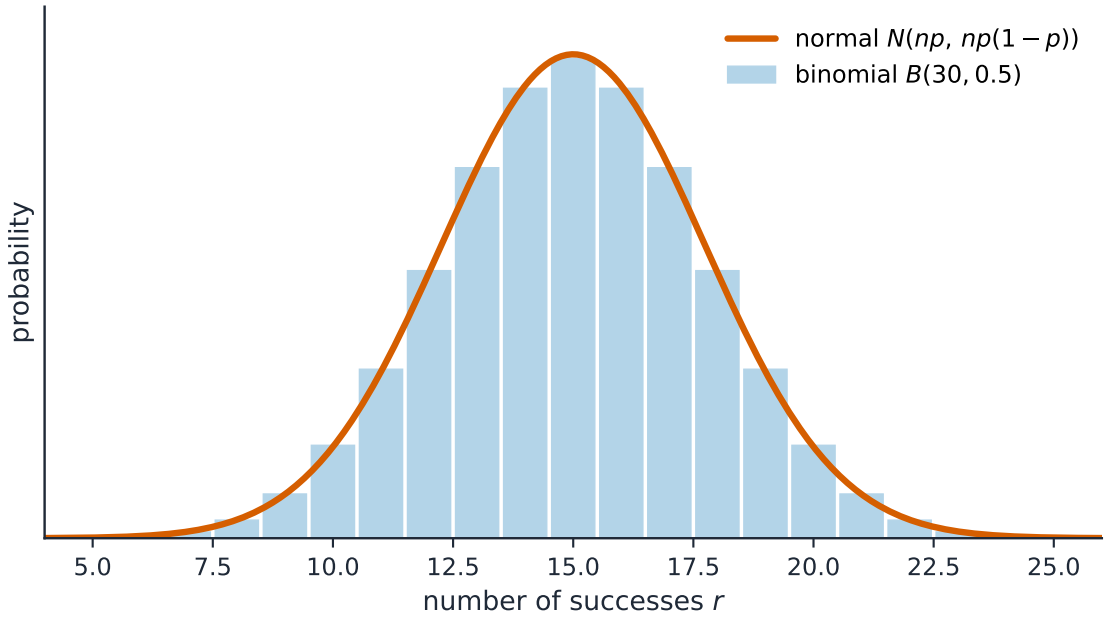
$$Z = \frac{X - \mu}{\sigma}.$$

A Galton board: balls fall through rows of pegs and pile up into the bell-shaped normal distribution Then $P(X < x) = P\left(Z < \dfrac{x-\mu}{\sigma}\right)$, which you read from the normal table Φ.



A normal probability is an area under the bell curve; standardizing rescales it to $Z \sim N(0,1)$.

The normal distribution is also a good **approximation** 近似 to the binomial when n is large. Because you replace a discrete variable by a continuous one, apply a **continuity correction** 连续性校正 (adjust by 0.5).



When n is large the binomial bars follow a normal curve of the same mean and variance.

Worked example. Bags of rice have mass $X \sim N(\mu, 0.14^2)$. Given that $P(X < 1.48) = 0.22$, find μ.

From the table, $P(Z < z) = 0.22$ gives $z = -0.772$. So

$$\frac{1.48 - \mu}{0.14} = -0.772 \ \Rightarrow\ \mu = 1.48 + 0.772 \times 0.14 = 1.59 \text{ kg (3 s.f.)}.$$