Motion in a circle

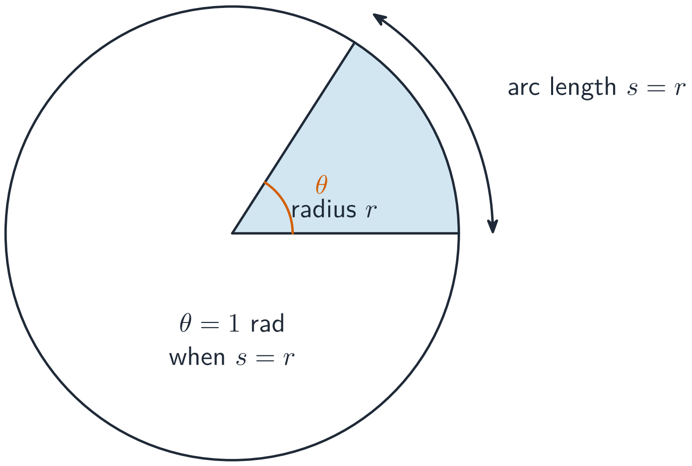
A-Level Physics

Angles in radians

The **radian** 弧度 is the angle made at the centre of a circle by an **arc** 弧 whose length equals the radius. For an arc of length s on a circle of radius r, the angle in radians is

$$\theta = \frac{s}{r}.$$

Radians have no unit (a ratio of lengths). A full circle has $s = 2\pi r$, so $\theta = 2\pi$ rad. A half-circle is π rad; a quarter is $\pi/2$ rad.



One radian is the angle whose arc length equals the radius

To convert: 1 rad = $180°/\pi \approx 57.3°$. Set your calculator to **radians** for this topic; ”degree” mode will give wrong answers.

Uniform circular motion: angular speed

A spinning fairground ride: every rider turns through the same angle each second.

An object moves in a circle of radius r at constant speed v. Define:

- **angular displacement** 角位移 θ —the angle (in radians) turned through by the radius from a chosen start line.
- **angular speed** 角速度 ω —the rate of change of angular displacement.

For uniform motion ω is constant and

$$\omega = \frac{\theta}{t}.$$

Unit: rad s^{-1}.

Period and frequency

If the object goes once round (2π rad, one **revolution** 圈) in time T (the **period** 周期), then

$$\omega = \frac{2\pi}{T} = 2\pi f,$$

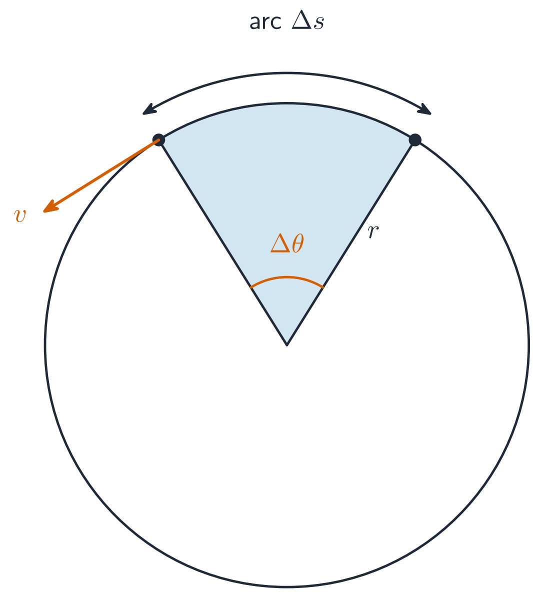
where $f = 1/T$ is the **frequency** 频率 of turning (Hz).

Linear and angular speed

In one period T the object travels a distance $2\pi r$ (the **circumference** 周长) at constant speed, so

$$v = \frac{2\pi r}{T} = r\omega.$$

This links the linear (**tangential 切向**) speed v with the angular speed ω. At a larger radius (for the same angular speed) the linear speed is larger —a child on the edge of a merry-go-round moves faster than one near the centre, even though both go round once in the same time.



As the radius turns through $\Delta\theta$ the object moves an arc Δs at speed v

Centripetal acceleration

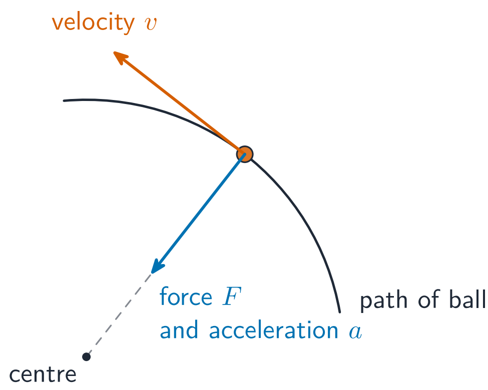
An object moving in a circle at constant speed still has a **changing velocity 速度**—its direction keeps changing, even though its size stays the same. A changing velocity needs an **acceleration 加速度**. This acceleration points **towards the centre** and is the **centripetal acceleration 向心加速度**.

Size

$$a = \frac{v^2}{r} = r\omega^2.$$

The two forms are equal because $v = r\omega$. Pick the one with the quantities you have.

The centripetal acceleration is **perpendicular 垂直** to the velocity at every instant — never along the direction of motion. (If part of it were along the motion, the speed would change.) Unit: m s^{-2}.



The velocity points along the tangent; the force and acceleration point to the centre

Centripetal force

A Ferris wheel: a centripetal force toward the centre keeps each car moving in a circle.

Image: Cayambe, CC BY-SA 3.0 (commons.wikimedia.org)

By Newton's second law, the resultant **force 力** on a body in circular motion at constant speed is

$$F = ma = \frac{mv^2}{r} = mr\omega^2.$$

This is the **centripetal force 向心力**. It **always points towards the centre** — perpendicular to the velocity.

The centripetal force is not a new kind of force —it is the **net result** of the real forces acting (tension, gravity, friction, electric attraction, normal contact force, ⋯). In a problem, work out which real force(s) provide it.

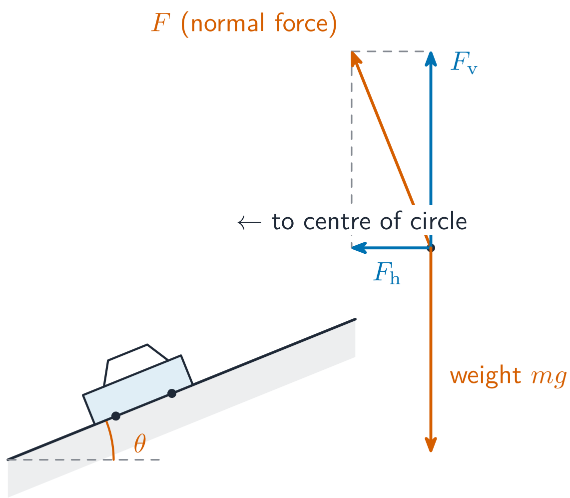
Where the centripetal force comes from

- **Ball on a string** in a horizontal circle: the **tension 张力** in the string.

- **Car turning a flat corner**: the **friction** 摩擦力 between tyres and road ($F = mv^2/r$). If the car goes too fast, friction is not enough and it skids outwards.
- **Banked corner** 倾斜 (no friction): the horizontal part of the **normal contact force** 支持力; $\tan\theta = v^2/(rg)$ for the angle that needs no friction.
- **Planet or satellite** 卫星 in **orbit** 轨道: the **gravitational attraction** 引力, $GMm/r^2 = mv^2/r$.
- **Electron** 电子 in a circular orbit (Bohr-style model): the **electrostatic** 静电 attraction between the electron and the positive **nucleus** 原子核:

$$\frac{kZe^2}{r^2} = \frac{m_e v^2}{r},$$

where $k = 1/(4\pi\varepsilon_0)$ and Z is the nuclear charge. Solve for v to get the orbital speed; then $T = 2\pi r/v$.

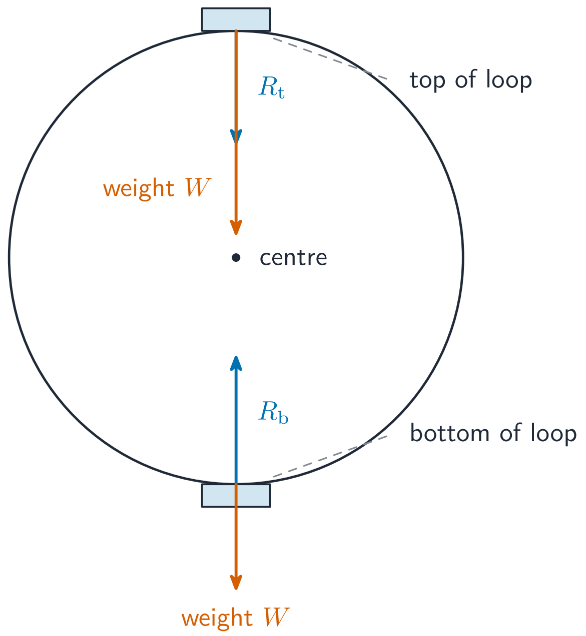


On a banked track the horizontal part of the road's force provides the centripetal force

Vertical circles

When the circle is upright, the speed is **not** constant (gravity does work) —but at each instant the **net force towards the centre** still equals mv^2/r:

- at the bottom of a loop: tension up, **weight** 重力 down, so $T - mg = mv^2/r$ —the tension is largest here.
- at the top of a loop: tension and weight both point down (towards the centre), so $T + mg = mv^2/r$ —the tension is smallest. For the slowest speed at the top with the string just tight, set $T = 0$: $mg = mv_{\text{min}}^2/r$, giving $v_{\text{min}} = \sqrt{gr}$.



Forces on a person at the top and bottom of a vertical circle

The constant-speed result ($v = r\omega$, ω constant) holds for horizontal circles, or where the force only bends the path (orbits in gravity, charges in a **magnetic field** 磁场).

How to structure a circular-motion answer

1. **Find the radius** r and choose v or ω. Use $v = r\omega$ to switch between them.
2. **Find the centripetal acceleration** with $a = v^2/r$ or $r\omega^2$.
3. **List the real forces** and write Newton's second law in the **radial** 径向 direction (towards the centre is positive). Set the net inward force equal to mv^2/r.
4. For **period or frequency**: use $\omega = 2\pi/T$, or $T = 2\pi r/v$.
5. Check the **directions**: centripetal force and acceleration point to the centre; the velocity is along the tangent.