Oscillations

A-Level Physics

Simple harmonic motion: definition

A pendulum clock keeps time using simple harmonic motion.

Image: Christoph Braun, CC0 (commons.wikimedia.org)

A particle moves with **simple harmonic motion** 简谐运动 (SHM) when its **acceleration** 加速度 is:

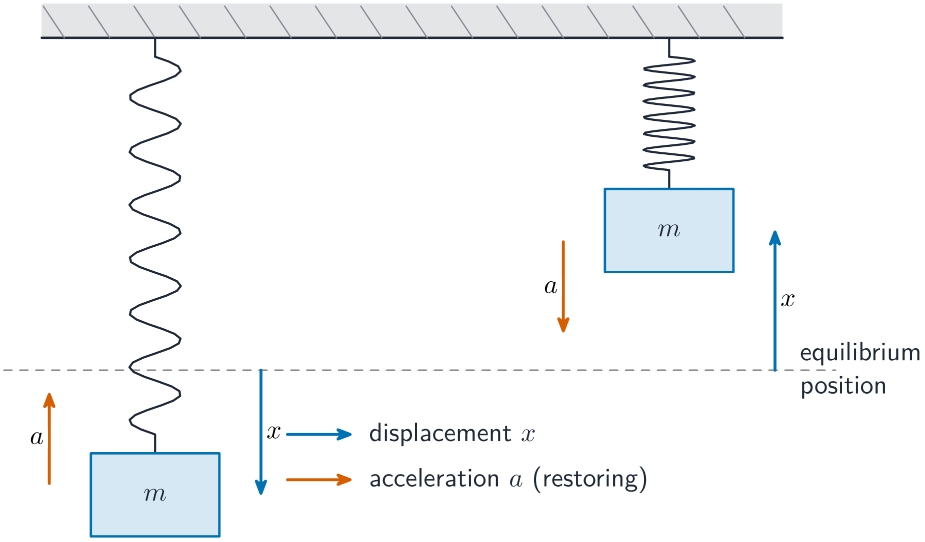
- **proportional to its displacement** from a fixed **equilibrium** 平衡 point, and
- **directed back towards that point** —opposite in sign to the displacement.

The defining equation is

$$a = -\omega^2 x,$$

where x is the **displacement** 位移 from equilibrium and ω is a positive constant, the **angular frequency** 角频率. The minus sign means ”directed back towards equilibrium”.

Many systems do this near a stable equilibrium: a mass on a **spring** 弹簧, a **pendulum** 单摆 (small swing), a floating block pushed down, the charge on a **capacitor** 电容器 in an LC circuit, atoms in a solid.



Acceleration always points back towards equilibrium, opposite to the displacement

Key terms

- **displacement** x —distance from equilibrium at a moment (a vector along the line of motion).
- **amplitude** 振幅 x_0 —the largest displacement from equilibrium. Always positive.
- **period** 周期 T —the time for one full oscillation.
- **frequency** 频率 f —the number of oscillations per second; $f = 1/T$. Unit: Hz.
- **angular frequency** ω —$\omega = 2\pi/T = 2\pi f$. Unit: rad s^{-1}.
- **phase difference** 相位差—the fraction of a cycle (in radians) by which one oscillation leads or lags another. A quarter-cycle apart is a phase difference of $\pi/2$.

So $T = 2\pi/\omega$ and $f = \omega/(2\pi)$ —given any one of ω, f, T you can find the others.

Displacement, velocity, acceleration in SHM

If the particle starts at $x = 0$ moving in the positive direction at $t = 0$, then

$$x = x_0 \sin(\omega t).$$

Differentiating once gives the **velocity** 速度:

$$v = x_0\omega \cos(\omega t) = v_0 \cos(\omega t),$$

where $v_0 = x_0\omega$ is the **maximum speed** (as the particle passes through equilibrium).

Differentiating again gives the acceleration:

$$a = -x_0\omega^2 \sin(\omega t) = -\omega^2 x,$$

which is the SHM defining equation again. (If the particle instead starts at the extreme position $x = x_0$ at $t = 0$, use $x = x_0 \cos(\omega t)$. Choose the one that fits the start conditions.)

Velocity in terms of displacement

A useful relation that does not use time:

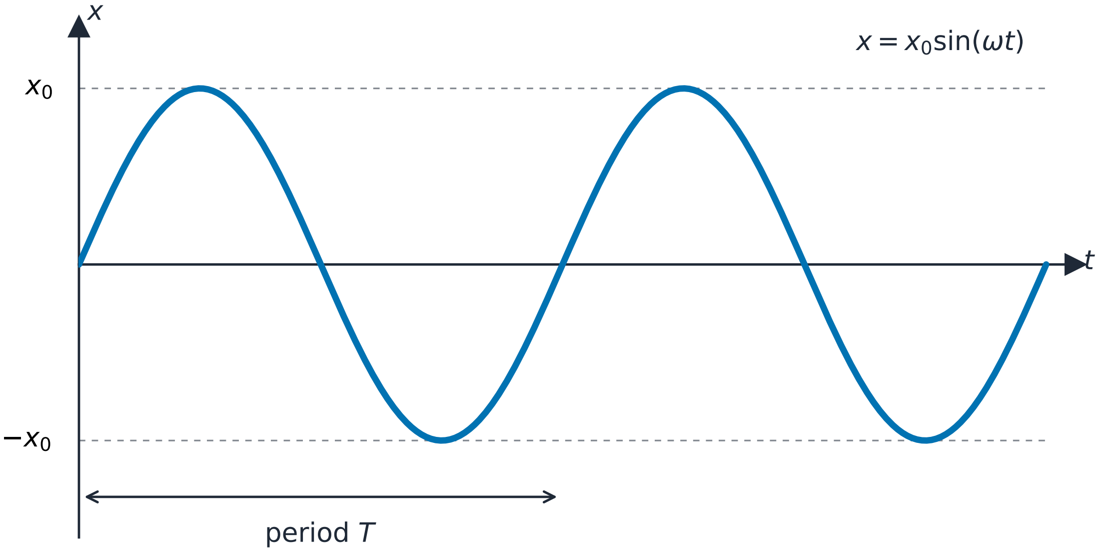
$$v = \pm\omega\sqrt{x_0^2 - x^2}.$$

- at equilibrium ($x = 0$): $v = \pm\omega x_0$ (maximum speed). Both signs, because the particle passes through equilibrium twice each cycle.
- at the extremes ($x = \pm x_0$): $v = 0$ (at rest for an instant).

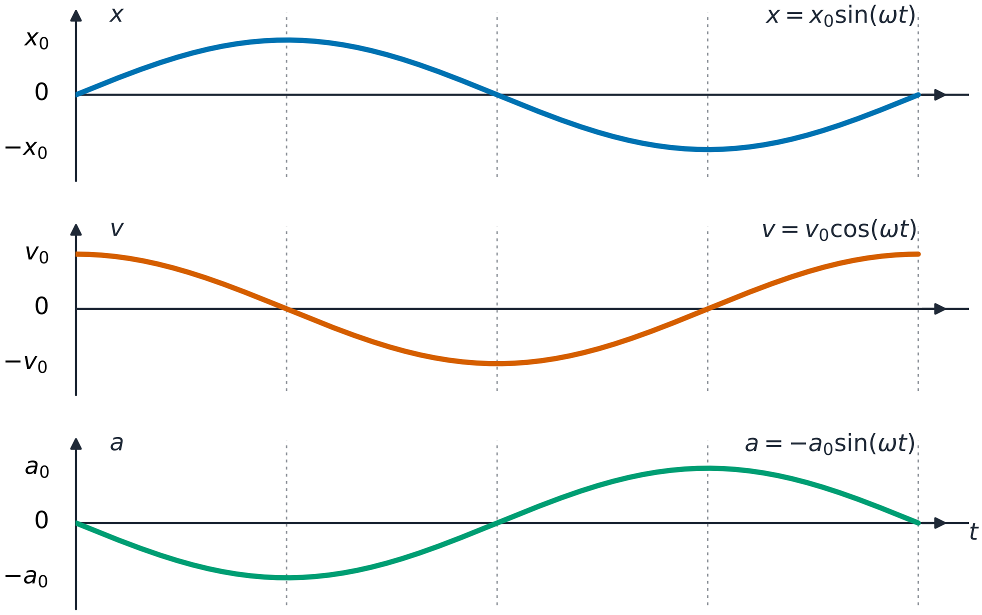
Graphs against time

For $x = x_0 \sin\omega t$:

- x vs t —a sine curve, amplitude x_0, period $T = 2\pi/\omega$.
- v vs t —a cosine curve, **leading x by** $\pi/2$, amplitude ωx_0.
- a vs t —a negative sine curve, **out of phase with x by** π (180°), amplitude $\omega^2 x_0$.



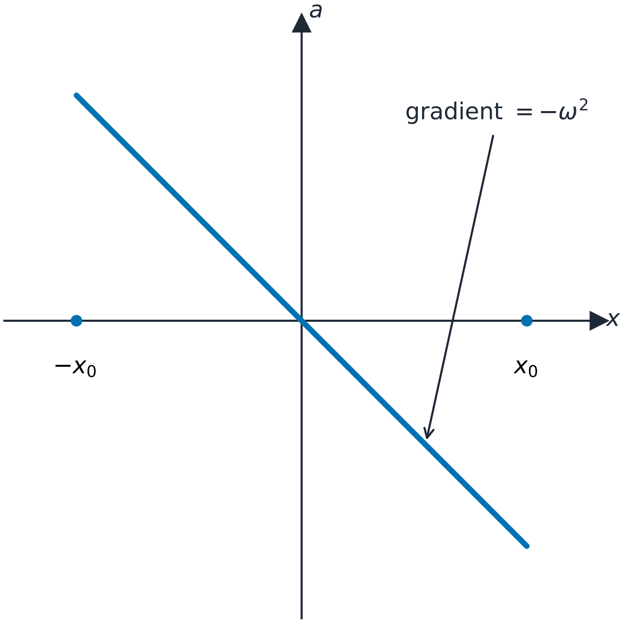
Displacement varies sinusoidally with time in simple harmonic motion



Velocity leads displacement by a quarter cycle; acceleration is exactly out of phase with displacement

Graph of a against x

A straight line through the origin with **negative gradient** $-\omega^2$. So you can read ω from the graph: gradient $= -\omega^2$, so $\omega = \sqrt{|\text{gradient}|}$, then $T = 2\pi/\omega$. This is a common exam pattern.



The acceleration–displacement graph is a straight line through the origin with gradient $-\omega^2$

Energy in simple harmonic motion

A simple harmonic oscillator keeps swapping energy between two forms:

- **kinetic energy 动能** $E_{\mathrm{K}} = \frac{1}{2}mv^2$.
- **potential energy** E_{P} (elastic for a spring, gravitational for a pendulum).

With no **damping 阻尼**, the **total energy** is constant (this is **conservation of energy** 能量守恒).

Maximum and minimum

- at **equilibrium** ($x = 0$): v is largest, so E_{K} is largest and E_{P} is smallest (zero, by choice).
- at the **extremes** ($x = \pm x_0$): $v = 0$, so $E_{\mathrm{K}} = 0$ and E_{P} is largest.

Total energy

Using $v_{\mathrm{max}} = \omega x_0$:

$$E_{\mathrm{total}} = \tfrac{1}{2}mv_{\mathrm{max}}^2 = \tfrac{1}{2}m\omega^2 x_0^2.$$

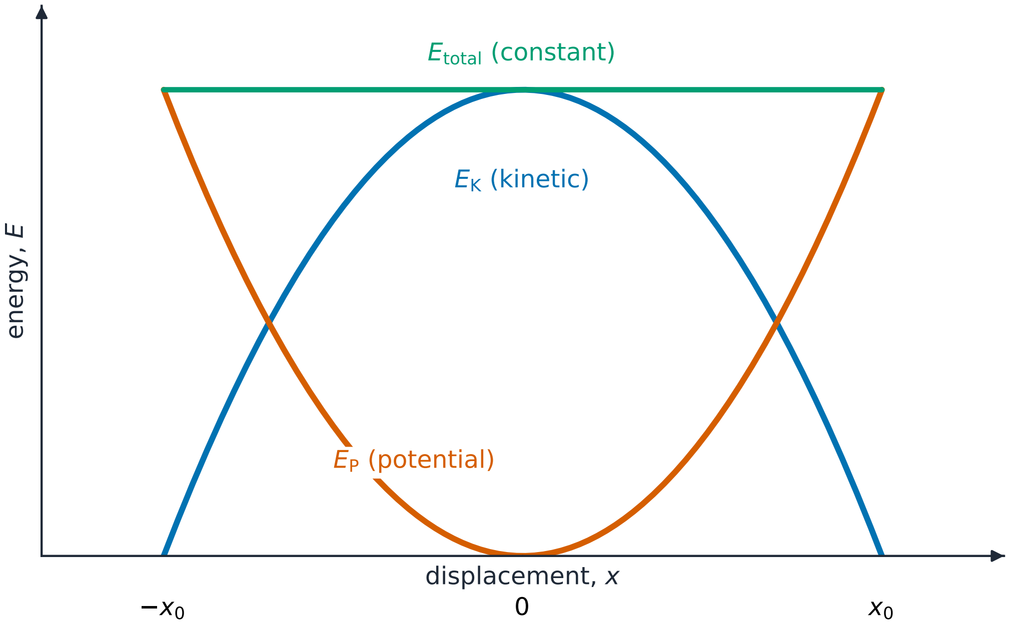
Two key facts: the total energy is proportional to the **square of the amplitude** (doubling x_0 gives four times the energy), and to ω^2.

Energy against displacement

Using $v^2 = \omega^2(x_0^2 - x^2)$:

$$E_{\mathrm{K}} = \tfrac{1}{2}m\omega^2(x_0^2 - x^2), \qquad E_{\mathrm{P}} = \tfrac{1}{2}m\omega^2 x^2.$$

So E_{K} is a downward parabola (peak at $x = 0$, zero at $x = \pm x_0$) and E_{P} is an upward parabola (zero at $x = 0$, largest at $x = \pm x_0$). Their sum is constant.



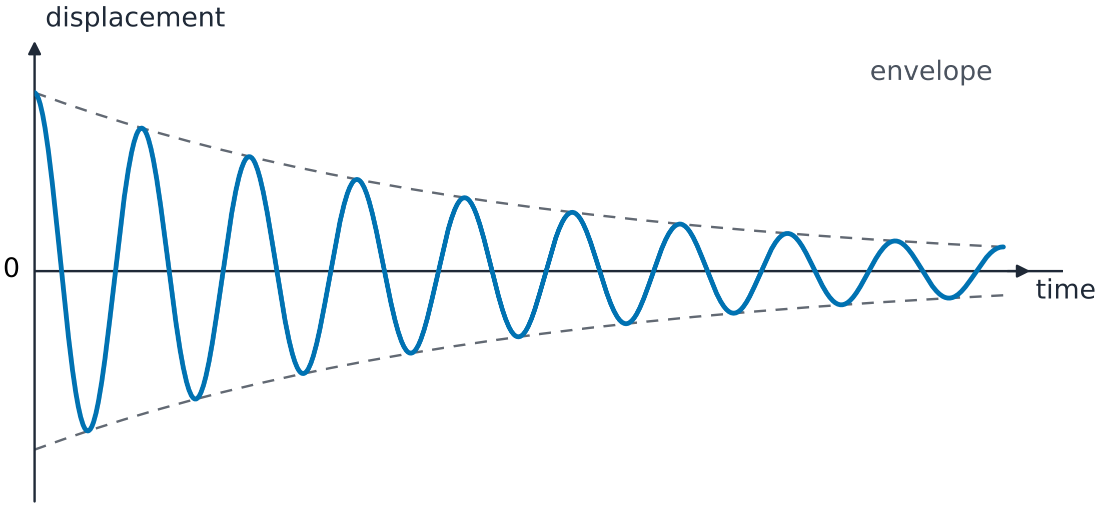
Kinetic and potential energy swap over a cycle while the total energy stays constant

Damped oscillations

A resistive force (**friction** 摩擦力, **drag** 阻力, **air resistance** 空气阻力) causes **damping** —the amplitude shrinks over time as energy is lost as heat. Three named cases:

Light damping

The amplitude **shrinks slowly** over many cycles (a **light damping** 轻阻尼 case). The system still oscillates near its natural frequency, but each cycle is smaller than the last. A car's suspension is light-to-medium damped, so bumps die away but the ride stays smooth.



In light damping the amplitude dies away slowly over many cycles

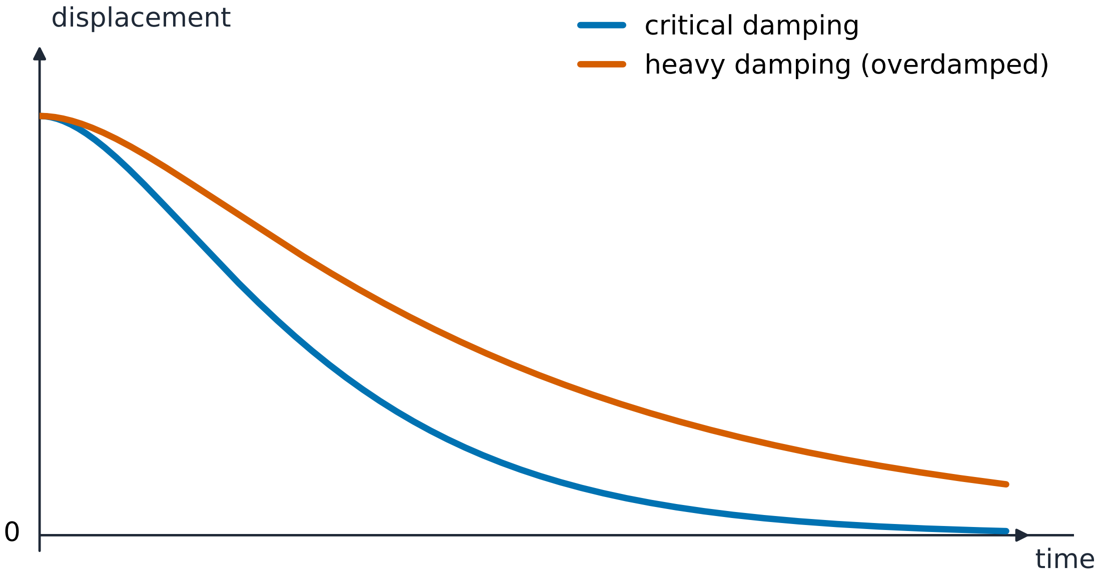
Critical damping

The least damping that brings the system back to equilibrium **without overshooting and without oscillating** —a **critical damping** 临界阻尼 case. It returns in the shortest time. A **galvanometer** 检流计 or analogue **voltmeter** 电压表 is critically damped so the needle settles quickly.

Heavy damping

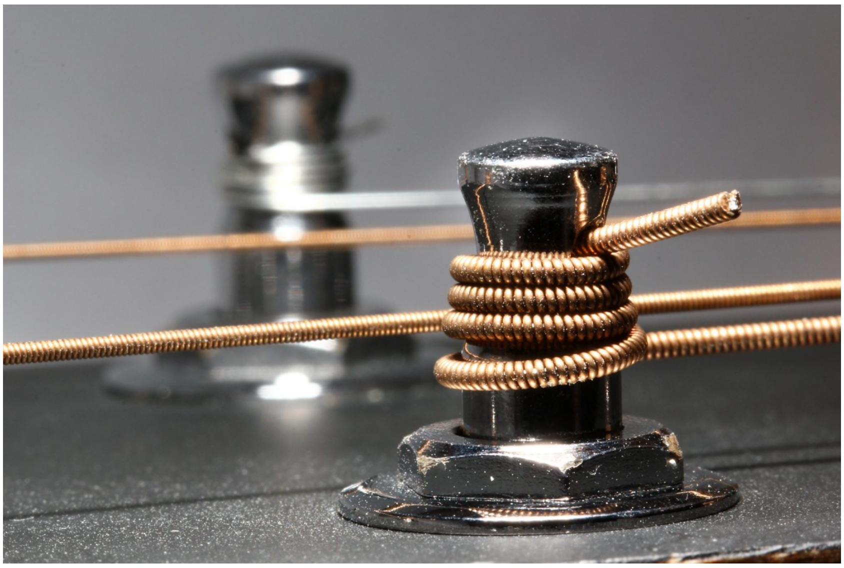
So much resistance that the system returns slowly, with no oscillation, but more slowly than the critical case —a **heavy damping** 过阻尼 case. A door with a strong closer is heavily damped.

On a displacement–time graph: light damping is a wave whose size dies away smoothly; critical damping returns quickly with no overshoot; heavy damping returns slowly.



Critical damping returns to equilibrium fastest without overshoot; overdamping returns more slowly

Forced oscillations and resonance



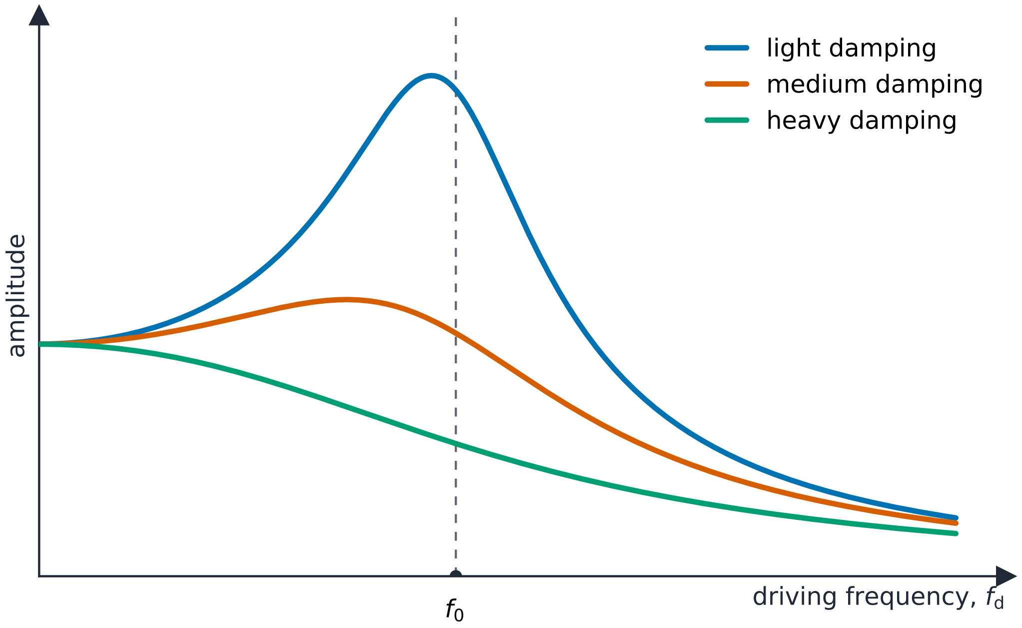
A plucked guitar string vibrates at its resonant frequencies.

Image: petur r, CC BY-SA 2.0 (commons.wikimedia.org)

A **forced oscillation** 受迫振动 is driven by an outside periodic force at a frequency f_d chosen by the experimenter. The system then oscillates at this **driving frequency** 驱动频率 f_d, not at its own natural frequency. A plot of amplitude against f_d is a **resonance curve** 共振曲线 with a peak.

Resonance

Resonance 共振 happens when the driving frequency equals the system's **natural frequency** 固有频率 f_0. At resonance the amplitude is **largest** and the energy transfer from the driver is most efficient.



The amplitude of a forced oscillation peaks at resonance, when the driving frequency equals the natural frequency

Examples:

- a swing pushed at the right rate builds up a large amplitude.
- a wine glass broken by a sound at its natural ringing frequency.
- a building shaken by an earthquake whose frequency matches a natural frequency —engineers design buildings so their natural frequencies avoid the main earthquake range.

The peak's shape depends on damping: **lighter damping → a sharper, higher peak**; **heavier damping → a broader, lower peak**, shifted slightly to lower frequency.