6.2 Approximating Areas with Riemann Sums

Name: ______________ Class: __________ Date: ____________ **Total: 12 marks**

Objective

Build the skills to answer exam questions on **Riemann sums** —estimating the area under a curve with rectangles.

You must be able to:

- compute **left**, **right**, and **midpoint** 中点 Riemann sums from a function or a table
- compute a **trapezoidal** 梯形 estimate
- decide whether an estimate **over-** or **under-estimates** using increasing/decreasing or concavity

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

Left and right sums

Estimate $\int_0^6 f\,dx$ with 3 equal subintervals of width $\Delta x = 2$, using the table:

x	**0**	**2**	**4**	**6**
$f(x)$	1	4	9	16

- **Left** sum uses the left endpoints: $2(1+4+9) = 28$.
- **Right** sum uses the right endpoints: $2(4+9+16) = 58$.

Midpoint sum

The **midpoint** sum uses the value at the middle of each subinterval. With $f(x) = x^2$ on $[0,6]$, $n = 3$, $\Delta x = 2$, midpoints $1, 3, 5$: $2\big(f(1)+f(3)+f(5)\big) = 2(1+9+25) = 70$.

Trapezoidal estimate

A **trapezoidal** sum averages each pair of endpoint heights: $\Delta x\left[\frac{f_0}{2} + f_1 + f_2 + \frac{f_3}{2}\right]$. For the table: $2\left[\frac{1}{2} + 4 + 9 + \frac{16}{2}\right] = 2(21.5) = 43$. It equals the average of the left and right sums.

■ **Over- or under-estimate**

For an **increasing** function, a left sum **under**-estimates and a right sum **over**-estimates. For a **concave-up** function, the trapezoidal sum **over**-estimates and the midpoint sum **under**-estimates.

2 Practice

Now apply the methods above.

Use the table for 2.1–2.3:

x	**0**	**3**	**6**	**9**
$g(x)$	2	5	6	10

2.1 Find the **left** Riemann sum for $\int_0^9 g\,dx$ with the three subintervals shown. [2]

2.2 Find the **right** Riemann sum for the same integral. [2]

2.3 Find the **trapezoidal** estimate. [2]

3 Exam-style questions

3.1 For an increasing function, a **left** Riemann sum gives an [1]

- **A** over-estimate
- **B** under-estimate
- **C** exact value
- **D** value that depends on concavity

3.2 A car's speed $v(t)$ in m/s is recorded:

t (s)	**0**	**4**	**8**	**12**
$v(t)$	0	12	20	24

(a) Estimate the distance travelled over $0 \le t \le 12$ using a **right** Riemann sum with three subintervals. [2]

(b) Is this an over- or under-estimate of the true distance? Give a reason. [1]

3.3 The function f is concave up on $[0, 4]$. A student estimates $\int_0^4 f\,dx$ with a trapezoidal sum. State, with a reason, whether the estimate is too large or too small. [2]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **6.2 Approximating Areas with Riemann Sums** lesson on the **Learn** page;
- read the **Approximating Areas with Riemann Sums** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $\Delta x = 3$; left $= 3(2 + 5 + 6) = 39$.

2.2 right $= 3(5 + 6 + 10) = 63$.

2.3 $3\left[\frac{2}{2} + 5 + 6 + \frac{10}{2}\right] = 3(17) = 51$. (Equivalently the average of 39 and 63.)

3.1 **B** —for an increasing function the left sum under-estimates.

3.2 (a) $4(12 + 20 + 24) = 224$ m. (b) Over-estimate —v is increasing, so right endpoints are the largest values on each subinterval.

3.3 Too large —for a concave-up curve each trapezoid lies **above** the curve, so the trapezoidal sum over-estimates.