

9.8 Finding the Area of a Polar Region

Name: _____ Class: _____ Date: _____

Total: 13 marks

Objective

Build the skills to answer exam questions on the **area of a polar region**.

You must be able to:

- apply $A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta$
- choose the correct θ -limits that trace the region once
- set up the area for a full curve or a sector

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The polar area formula

The area swept by $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

The region is a fan of thin triangular sectors of area $\frac{1}{2}r^2 d\theta$.

■ A worked area

The circle $r = 2$: $A = \frac{1}{2} \int_0^{2\pi} 2^2 d\theta = \frac{1}{2}(4)(2\pi) = 4\pi$ —matching πr^2 .

■ Choosing the limits

For $r = 2 \cos \theta$ (a circle traced once as θ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$), integrate over exactly that range so the curve is traced **once**.

■ One petal of a rose

For $r = \sin(2\theta)$, one petal is traced as θ goes from 0 to $\frac{\pi}{2}$. Its area is $\frac{1}{2} \int_0^{\pi/2} \sin^2(2\theta) d\theta$.

2 Practice

Now apply the methods above.

2.1 State the polar area formula. [1]

2.2 Set up the area enclosed by $r = 3$ (a full circle). [2]

2.3 Evaluate $\frac{1}{2} \int_0^{2\pi} 9 \, d\theta$. [2]

3 Exam-style questions

3.1 The area of a polar region is [1]

- **A** $\int_{\alpha}^{\beta} f(\theta) \, d\theta$
 - **B** $\frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta$
 - **C** $\pi \int_{\alpha}^{\beta} f(\theta) \, d\theta$
 - **D** $\frac{1}{2} \int_{\alpha}^{\beta} f(\theta) \, d\theta$
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3.2 Consider $r = 1 + \cos \theta$ (a cardioid), traced once for $0 \leq \theta \leq 2\pi$.

(a) Set up the area integral. [2]

(b) Expand $(1 + \cos \theta)^2$ ready to integrate. [2]

3.3 One petal of the rose $r = 2 \sin(3\theta)$ is traced as θ goes from 0 to $\frac{\pi}{3}$. Set up the area integral for one petal. [3]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **9.8 Finding the Area of a Polar Region** lesson on the **Learn** page;
- read the **Finding the Area of a Polar Region** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$

2.2 $A = \frac{1}{2} \int_0^{2\pi} 3^2 d\theta.$

2.3 $\frac{1}{2}(9)(2\pi) = 9\pi.$

3.1 B —half the integral of r^2 .

3.2 (a) $A = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta.$ (b) $(1 + \cos \theta)^2 = 1 + 2 \cos \theta + \cos^2 \theta.$

3.3 $A = \frac{1}{2} \int_0^{\pi/3} (2 \sin(3\theta))^2 d\theta = \frac{1}{2} \int_0^{\pi/3} 4 \sin^2(3\theta) d\theta.$