Limits and Continuity

AP Calculus BC

Introducing Calculus: Can Change Occur at an Instant?

Calculus is the mathematics of **change** 变化 and of **accumulation** 累积. It answers two big questions: how fast is something changing *right now*, and how much has piled up *so far*? Unit 1 builds the one tool both questions rest on –the **limit** 极限.

Start with a puzzle. A car's speedometer reads 60 km/h. What does that mean *at a single instant* 瞬间? Speed is distance over time. But at one instant no time passes and no distance is covered, so the fraction looks like $\frac{0}{0}$ –undefined.

- The **average rate of change** 平均变化率 uses a whole *interval* 区间: the change in one quantity divided by the change in another. It divides by zero, and so is undefined, when the change in the input would be zero.
- The **instantaneous rate of change** 瞬时变化率 is what we want *at a point*. It is the value the average rate **approaches** 趋近 as the interval shrinks toward zero length.

The clever move is not to plug in zero (undefined), but to watch what the average rate *approaches* as the interval gets smaller and smaller. That approaching value is a **limit**. So calculus lets us describe change at an instant –as a limit of average rates over ever-shorter intervals. This one idea powers the **derivative** 导数 (Unit 2) and, run in reverse, the **integral** 积分 (Unit 6). Everything else in this unit defines limits carefully and computes them reliably.

Defining Limits and Using Limit Notation

Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made **arbitrarily** 任意地 close to R by taking x **sufficiently** 足够 close to c –**but not equal to** c. We write

$$\lim_{x \to c} f(x) = R$$

and read it: "the limit of $f(x)$, as x approaches c, equals R."

The last words are the heart of a limit: it describes the **behavior** 行为 of f *near* c, not the value *at* c. The function may be undefined at c, or defined but equal to something else –the limit does not care.

A limit can be shown in three ways: **graphically** 用图象, **numerically** 用数值 (a table), and **analytically** 用解析式 (algebra). Learning to move between these representations is a core skill.

*(Note: the epsilon-delta definition of a limit is **not** tested on the AP Exam, so this handout does not use it.)*

Estimating Limit Values from Graphs

A graph is often the fastest way to read a limit. To find $\lim_{x \to c} f(x)$, run your finger along the curve toward $x = c$ from each side and ask: what height is the curve heading for?

- Trace from the **left** (inputs smaller than c): this gives the **left-hand limit** 左极限, $\lim_{x \to c^-} f(x)$.
- Trace from the **right** (inputs larger than c): this gives the **right-hand limit** 右极限, $\lim_{x \to c^+} f(x)$.
- These are the **one-sided limits** 单侧极限. If both head to the *same* height R, then the two-sided limit exists and $\lim_{x \to c} f(x) = R$.

Crucially, **ignore the point itself**. Graphs mark the difference between the limit and the value:

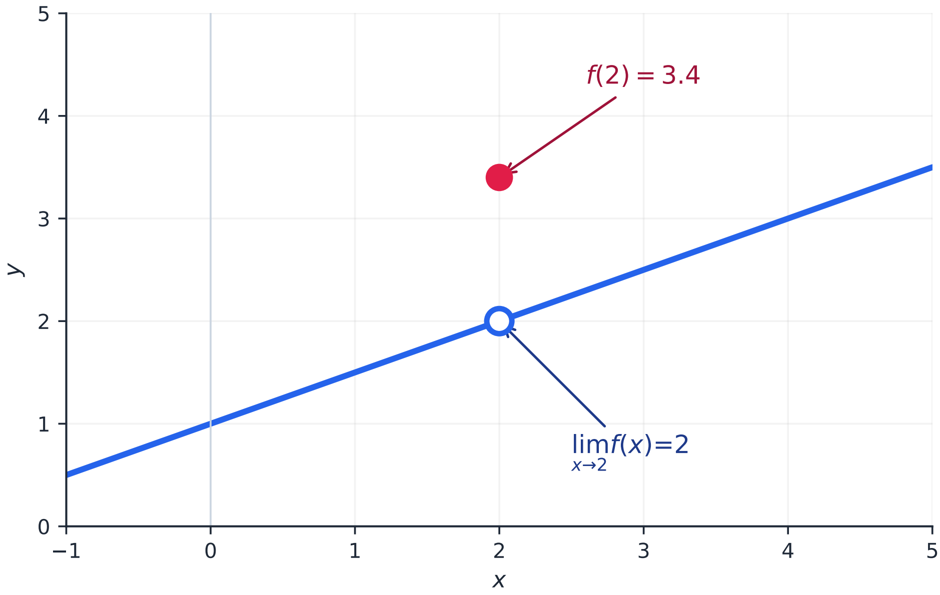
- An **open circle** 空心圆 marks a height the curve *approaches* but does not reach –a ”hole” 空洞.
- A **closed circle** 实心圆 marks the actual value $f(c)$.

So a curve may approach $R = 3$ from both sides (limit is 3) while a filled dot sits at height 5 (value $f(c) = 5$). The limit is 3; the two need not match.

A limit **does not exist** (often written **DNE**) when the two sides disagree (a **jump** 跳跃), when the function is **unbounded** 无界 (grows without limit), or when it **oscillates** 振荡 forever near c. For example:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty, \qquad \lim_{x \to 0} \frac{|x|}{x} \text{ DNE}, \qquad \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ DNE}.$$

Watch the **scale** 比例 of a graph: a zoomed-out picture can hide important behavior near a point, so confirm with algebra when you can.



The open circle is the height the curve approaches (the limit); the filled dot is the actual value $f(2)$ – they need not agree.

Estimating Limit Values from Tables

When you have data or a formula but no picture, a **table** 表格 of values estimates a limit numerically. Choose inputs that creep toward c from both sides and watch the outputs.

For example, to estimate $\lim_{x\to 2} \frac{x^2-4}{x-2}$ (which is $\frac{0}{0}$ at $x=2$):

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4, so we estimate the limit is 4. A table only *suggests* a value –it is a numerical estimate, not a proof.

Determining Limits Using Algebraic Properties of Limits

Most limits are found **analytically** using **limit theorems** 极限定理. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, the limit of a combination is the same combination of the limits:

- **Sum / difference:** $\lim_{x\to c}\left[f(x) \pm g(x)\right] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
- **Product:** $\lim_{x\to c}\left[f(x)\,g(x)\right] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
- **Quotient:** $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided the bottom limit is not 0.
- **Composite** 复合函数: if g is continuous at $\lim_{x\to c} f(x)$, then $\lim_{x\to c} g\big(f(x)\big) =$

$g\Big(\lim_{x\to c} f(x)\Big)$.

The practical rule: for a function built from polynomials, roots, and the like, first try **direct substitution** 直接代入–put $x = c$ in. If you get a real number, that is the limit. One-sided limits obey the same theorems, read from one direction only.

Determining Limits Using Algebraic Manipulation

Direct substitution sometimes gives the **indeterminate form** 未定式 $\frac{0}{0}$. This does **not** mean the limit fails –it means you must rewrite the function into an **equivalent form** 等价形式 that removes the trouble, then substitute. Three standard moves:

- **Factor and cancel** 因式分解并约分 a **rational function** 有理函数. Example: $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$.
- Multiply by the **conjugate** 共轭 to simplify a **radical** 根式. Example: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{x}{x\big(\sqrt{x+1}+1\big)} = \frac{1}{2}$.
- **Use alternate forms** of trigonometric functions (identities) to simplify.

The cancelled factor is why the original graph had a hole: the two functions agree everywhere except at $x = c$, so they share the same limit there.

Selecting Procedures for Determining Limits

This is a skill topic, not new content: choose the right tool for the limit in front of you.

1. **Try direct substitution first.** A real answer means you are done.
2. Getting $\frac{0}{0}$? **Rewrite** –factor and cancel, or use the conjugate, or a trig identity – then substitute.
3. A **non-zero number over** 0 (like $\frac{5}{0}$)? The limit is infinite or DNE –check the sign from each side (see vertical asymptotes below).
4. As $x \to \pm\infty$? Compare the **dominant** 主导 terms (see limits at infinity).
5. Trapped between two functions? The **squeeze theorem** may apply.

Determining Limits Using the Squeeze Theorem

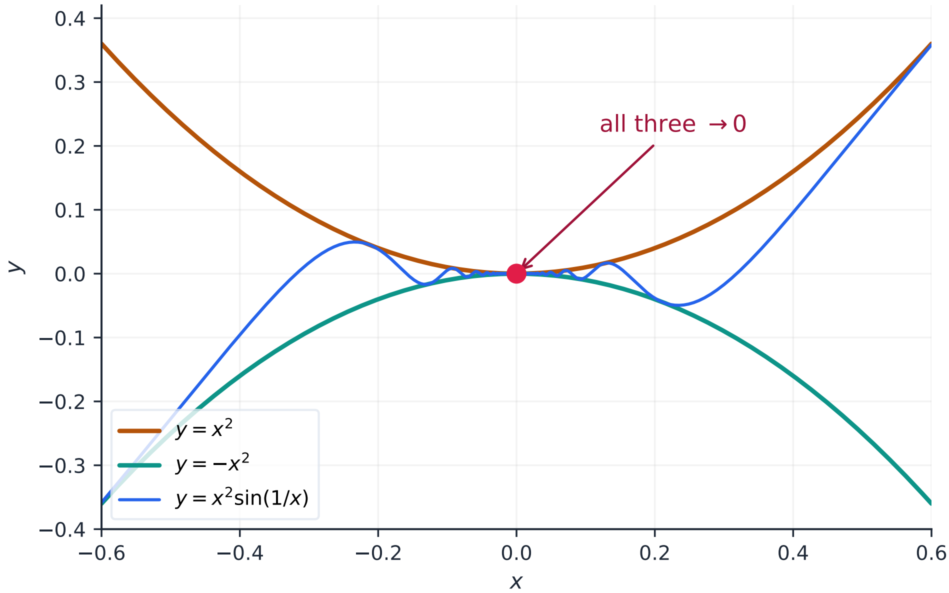
The **squeeze theorem** 夹逼定理 (also called the sandwich theorem) finds a limit by trapping the function between two others. If $g(x) \le f(x) \le h(x)$ near c, and

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L,$$

then f is squeezed to the same place: $\lim_{x\to c} f(x) = L$.

The two famous results proved this way, both used throughout calculus, are:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$



The Squeeze Theorem traps x squared sin(1/x) between minus x squared and x squared

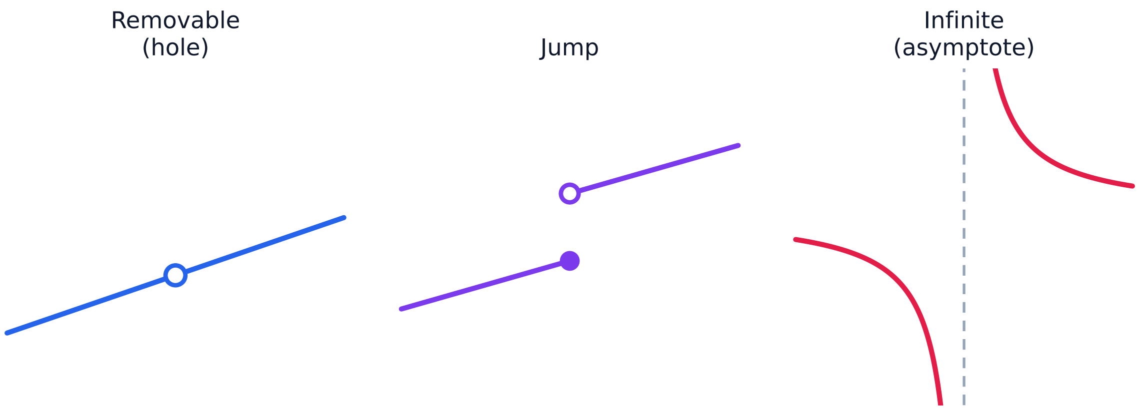
Connecting Multiple Representations of Limits

Another skill topic: the same limit lives in a graph, a table, and an algebraic form, and you should be able to translate between them. A graph shows the shape and any holes or jumps; a table gives numerical evidence; algebra gives an exact value and a reason. Strong answers use one representation to *confirm* another.

Exploring Types of Discontinuities

A function is **discontinuous** 间断 at c when its graph "breaks" there. There are three types:

- **Removable discontinuity** 可去间断–a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity** 跳跃间断–the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity** 无穷间断–the function blows up to $\pm\infty$ at a **vertical asymptote** 垂直渐近线.



The three types of discontinuity: removable, jump, and infinite

Defining Continuity at a Point

Continuity is defined by a three-part test. A function f is **continuous 连续** at $x = c$ exactly when all three hold:

$$\boxed{f(c) \text{ exists} \quad \text{and} \quad \lim_{x \to c} f(x) \text{ exists} \quad \text{and} \quad \lim_{x \to c} f(x) = f(c)}$$

In words: the point is there, the limit is there, and the two agree. If any one fails, f is discontinuous at c. This test is the backbone of nearly every continuity question, so learn it as a checklist.

Confirming Continuity over an Interval

A function is **continuous on an interval 在区间上连续** if it is continuous at *every* point of that interval. You rarely check point by point, because whole families are continuous on their domains:

> **Polynomial, rational, power, exponential 指数, logarithmic 对数, and trigonometric 三角 functions are continuous at every point of their domains.**

So a rational function is continuous everywhere *except* where its denominator is zero; $\ln x$ is continuous for $x > 0$; and so on. Knowing this lets you declare continuity quickly and correctly.

Removing Discontinuities

If the **limit exists** at a hole, the discontinuity is **removable**: redefine the function at that one point to equal the limit, and the graph is repaired. Formally, set the missing value to $\lim_{x \to c} f(x)$.

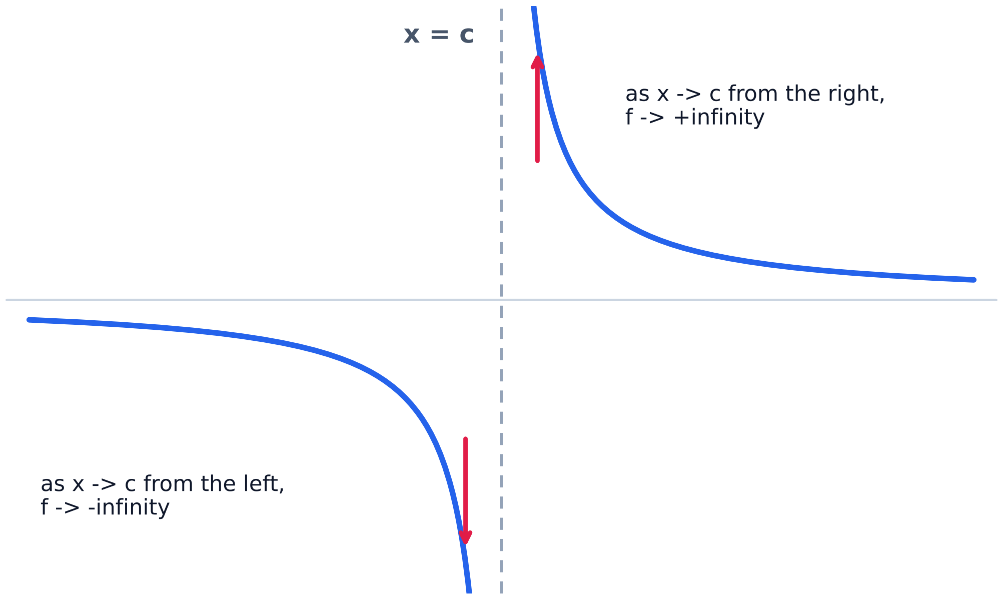
For a **piecewise-defined function 分段函数**, continuity at a boundary $x = c$ needs the two pieces to *meet*: the left piece's value, the right piece's value, and $f(c)$ must all be equal. This is a common exam setup –you solve for a **parameter** 参数 (an unknown constant) that makes the pieces match:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

Connecting Infinite Limits and Vertical Asymptotes

The idea of a limit extends to **infinite limits** 无穷极限. When a function grows without bound near $x = c$, we write $\lim_{x \to c} f(x) = \pm\infty$. This describes a **vertical asymptote** at $x = c$: the graph hugs the vertical line $x = c$ and shoots off toward $\pm\infty$.

This happens where a **non-zero number is divided by something approaching** 0, such as at a zero of a denominator that does not cancel. Always check each side separately –the two sides can shoot **opposite** ways (one to $+\infty$, one to $-\infty$).



Near a vertical asymptote the graph hugs the line $x = c$, and the two sides can shoot to opposite infinities.

Connecting Limits at Infinity and Horizontal Asymptotes

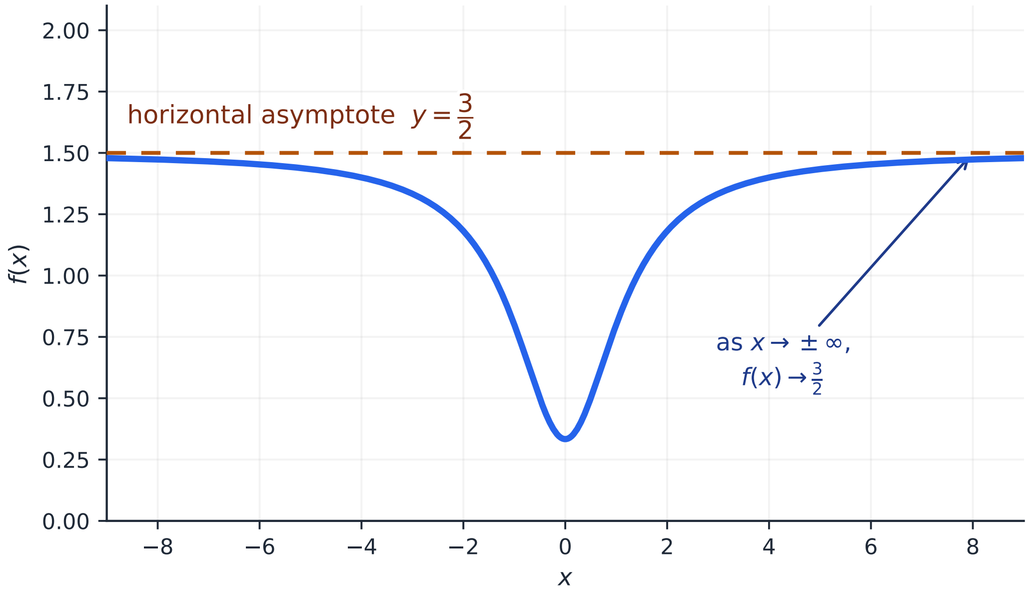
We can also let the **input** grow: **limits at infinity** 无穷远处的极限 describe the **end behavior** 末端行为 of a function as $x \to \pm\infty$. If the outputs settle toward a finite value L, then $y = L$ is a **horizontal asymptote** 水平渐近线.

For a rational function, compare the **degrees** 次数 of the top and bottom:

- top degree $<$ bottom degree $\Rightarrow$ limit is 0 (asymptote $y = 0$);
- top degree $=$ bottom degree $\Rightarrow$ limit is the **ratio of the leading coefficients** 首项系数之比;

- top degree $>$ bottom degree $\Rightarrow$ the function is unbounded (no horizontal asymptote).

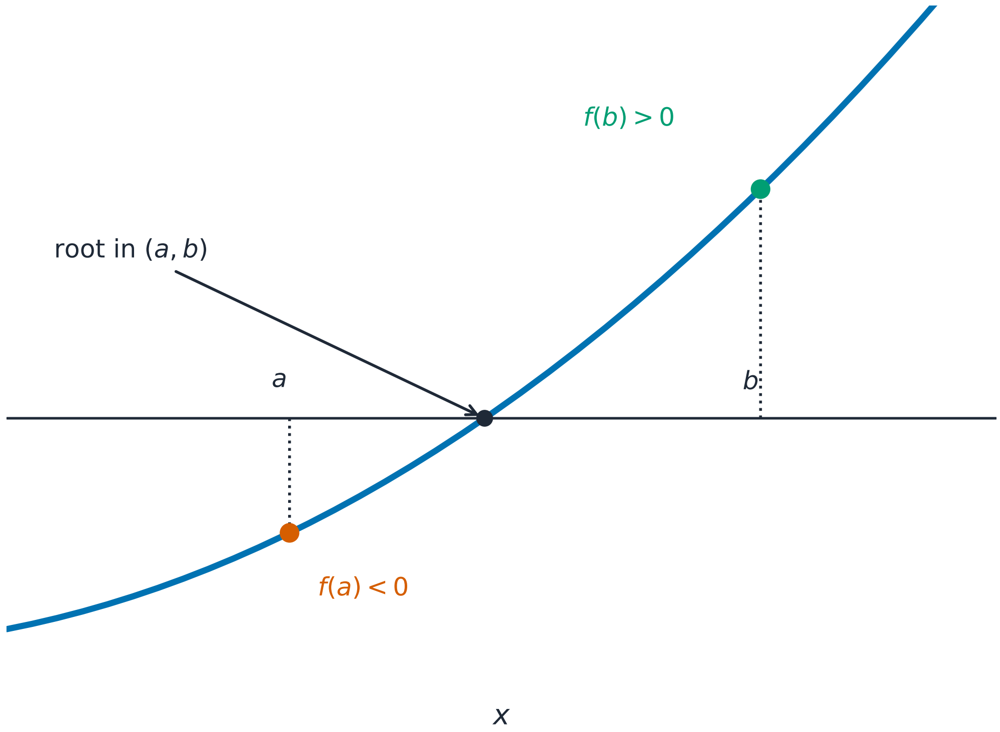
More generally, we compare the **relative magnitudes** 相对大小 (relative growth rates) of functions: far out, an exponential beats any polynomial, and a polynomial beats any logarithm. On the exam, "as $t \to \infty$, which quantity is larger/where does the rate settle?" is answered with a limit at infinity.



A limit at infinity produces a horizontal asymptote

Working with the Intermediate Value Theorem (IVT)

The **Intermediate Value Theorem** 介值定理 is an **existence theorem** 存在性定理–it guarantees a value exists without telling you where:



Opposite signs of f(a) and f(b) trap a root between a and b

If f is **continuous** on the closed interval $[a, b]$, and d is any number **between** $f(a)$ and $f(b)$, then there is **at least one** number c in (a, b) with $f(c) = d$.

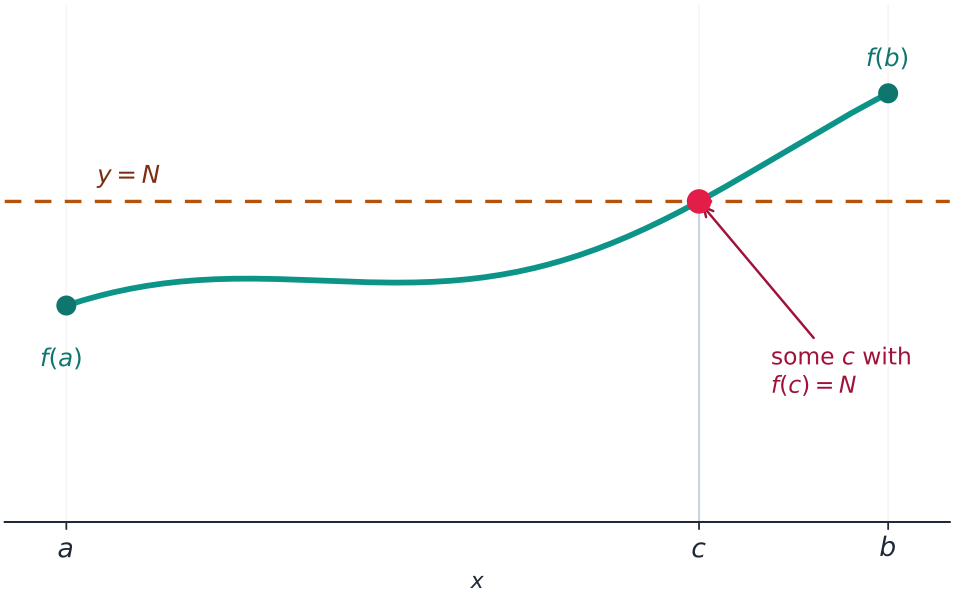
An unbroken curve cannot skip a height between its endpoints –it must pass through every one.

Exam skill –how to justify with the IVT. These questions appear almost every year (for example, "Must there be a value c with $R(c) = 155$?" or "Is there a time when $r'(t) = -6$?"). A full-credit justification has three moves:

1. **State continuity.** Say the function is continuous on $[a, b]$ (often because it is differentiable, or given continuous).
2. **Show d is trapped.** Compute the two endpoint values and show the target d lies **between** them, e.g. $f(a) < d < f(b)$.
3. **Conclude by name.** "By the Intermediate Value Theorem, there is a c in (a, b) with $f(c) = d$."

Skipping the continuity statement, or not showing d is between the endpoints, loses the point –the theorem *requires* both conditions.

Worked example. Evaluate $\lim_{x\to\infty} \dfrac{3x^2 - 5}{2x^2 + x}$. Divide top and bottom by the highest power, x^2: $\dfrac{3 - 5/x^2}{2 + 1/x} \to \dfrac{3 - 0}{2 + 0} = \dfrac{3}{2}$. Because the limit is a finite number, the line $y = \frac{3}{2}$ is a **horizontal asymptote** of the graph.



A continuous curve from $(a, f(a))$ to $(b, f(b))$ must cross every height d in between at least once.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ —the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes where the denominator (not the numerator) is zero.