Parametric Equations, Polar Coordinates, and Vector-Valued Functions

AP Calculus BC

Defining and Differentiating Parametric Equations

Parametric equations 参数方程 give x and y each as functions of a **parameter** t (often time): $x = x(t)$, $y = y(t)$. They trace a curve that need not be a function of x. The slope of the curve is found with the chain rule:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad (dx/dt \neq 0).$$

Worked example. For $x = t^2$, $y = t^3 - t$, find the slope at $t = 2$. Here $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 1$, so $\frac{dy}{dx} = \frac{3t^2 - 1}{2t}$; at $t = 2$ this is $\frac{11}{4}$.

Second Derivatives of Parametric Equations

The second derivative is **not** $\frac{d^2y/dt^2}{d^2x/dt^2}$. Instead, differentiate the first derivative with respect to t, then divide by dx/dt again:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

Use it to test concavity of a parametric curve.

Finding Arc Lengths of Curves Given by Parametric Equations

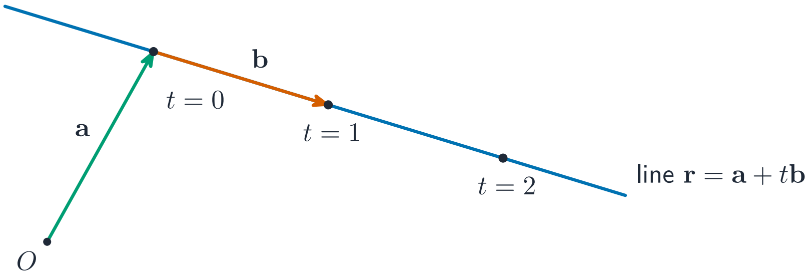
The length of a parametric curve for t from a to b is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This is the parametric version of arc length –summing tiny hypotenuses $\sqrt{dx^2 + dy^2}$ over the parameter.

Defining and Differentiating Vector-Valued Functions

A **vector-valued function** 向量值函数 $\vec{r}(t) = \langle x(t), y(t)\rangle$ gives a position vector for each t. Differentiate **component-wise**: the **velocity** is $\vec{v}(t) = \langle x'(t), y'(t)\rangle$ and the **acceleration** is $\vec{a}(t) = \langle x''(t), y''(t)\rangle$. The **speed** is the magnitude $|\vec{v}| = \sqrt{x'^2 + y'^2}$.



A vector-valued line: start at a, then slide by t lots of the direction b

Integrating Vector-Valued Functions

Integrate a vector function **component by component**. Given acceleration or velocity plus an initial condition, integrate each component and use the condition to find the constants –recovering velocity from acceleration, or position from velocity.

Solving Motion Problems Using Parametric and Vector-Valued Functions

For a particle moving in a plane: position is $\langle x(t), y(t)\rangle$, velocity and acceleration are its derivatives, speed is $|\vec{v}|$, and the **distance traveled** over $[a, b]$ is

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

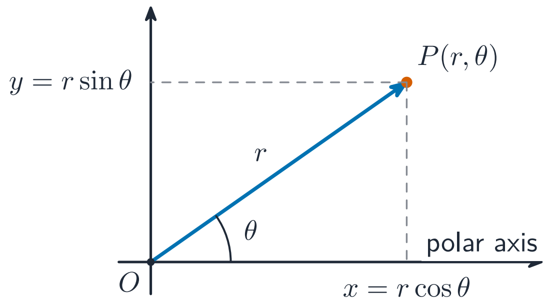
Exam skill: these plane-motion problems appear on the BC free-response nearly every year –be fluent finding speed, the position at a later time (initial point plus the integral of velocity), and total distance.

Worked example. A particle has position $\langle t^2,\ t^3 - t\rangle$. Its velocity is $\langle 2t,\ 3t^2 - 1\rangle$, so at $t = 1$ the velocity is $\langle 2,\ 2\rangle$ and the speed is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. Its position at $t = 2$ is $\langle 4,\ 6\rangle$.

Defining Polar Coordinates and Differentiating in Polar Form

Polar coordinates 极坐标 locate a point by its distance r from the origin and angle θ: convert with $x = r\cos\theta$, $y = r\sin\theta$. A polar curve $r = f(\theta)$ is a parametric curve in θ, so its slope is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}, \quad \text{with } x = r\cos\theta,\ y = r\sin\theta.$$



Polar coordinates give a point by its distance r and angle theta

Finding the Area of a Polar Region

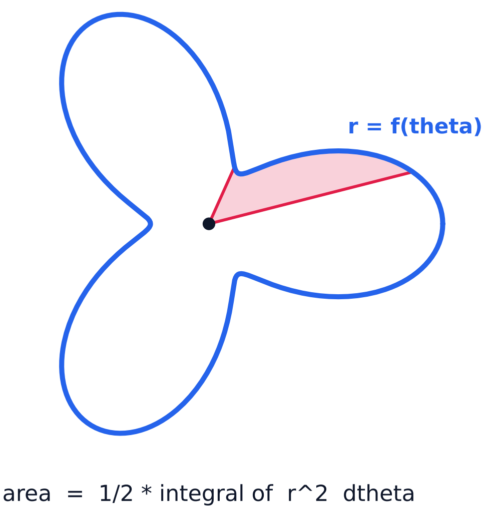
The area swept out by a polar curve $r = f(\theta)$ from α to β is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta.$$

The region is a "fan" of thin triangular sectors; choosing the correct θ-limits (where the curve starts and finishes tracing the region) is the main challenge.

Worked example. Find the area of one petal of the rose $r = 2\sin(2\theta)$ (traced for θ from 0 to $\frac{\pi}{2}$). Using $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$,

$$A = \frac{1}{2}\int_{0}^{\pi/2} (2\sin 2\theta)^2 \, d\theta = \int_{0}^{\pi/2} (1 - \cos 4\theta)\, d\theta = \big[\theta - \tfrac{\sin 4\theta}{4}\big]_{0}^{\pi/2} = \frac{\pi}{2}.$$



The shaded region is a fan of thin sectors swept from α to β; each has area $\frac{1}{2}r^2\,d\theta$, so the total is $\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$.

Finding the Area of the Region Bounded by Two Polar Curves

For the area between an outer curve r_1 and an inner curve r_2, subtract the sectors:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right) d\theta.$$

Find the **intersection angles** first (set $r_1 = r_2$), and be careful which curve is outer over each interval –they can swap.

Exam tips

- For **parametric** curves, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.
- A **vector-valued** function carries the same information —differentiate/integrate it component by component.
- In **polar**, convert with $x = r\cos\theta$, $y = r\sin\theta$; area swept is $\frac{1}{2}\int r^2\, d\theta$.
- Watch the direction of tracing (the sign of dx/dt) and set correct θ-limits for polar area.
- Eliminate the parameter to recover the ordinary y-vs-x shape when it helps.