# Torque and Rotational Dynamics

AP Physics C: Mechanics

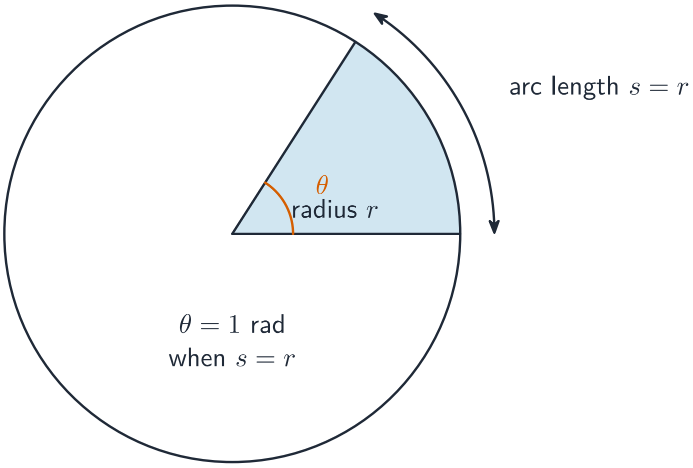
## Rotational Kinematics

Rotation mirrors linear motion with angular quantities, defined as derivatives:

$$\omega = \frac{d\theta}{dt}, \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Here $\theta$ is **angular displacement** 角位移 in **radians** 弧度, $\omega$ the **angular velocity** 角速度, and $\alpha$ the **angular acceleration** 角加速度. (AP works with their magnitudes and signs; the 3D vector directions are not assessed.) For constant $\alpha$, the kinematic equations are the linear ones re-lettered:

$$\omega = \omega_0 + \alpha t, \qquad \Delta\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2, \qquad \omega^2 = \omega_0^2 + 2\alpha\,\Delta\theta.$$




*One radian is the angle whose arc length equals the radius*

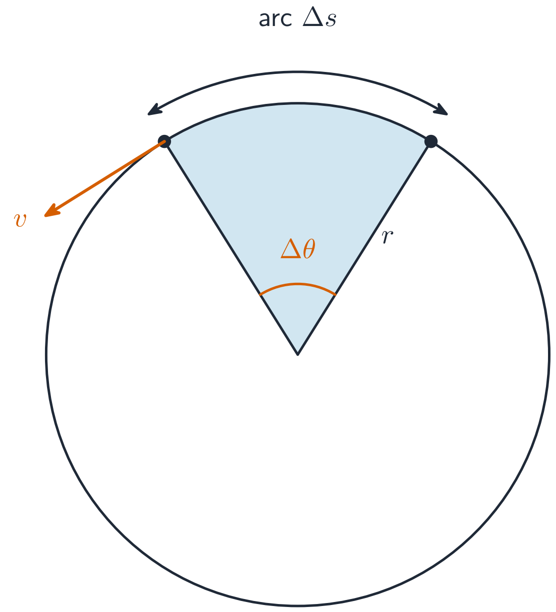
**Worked example.** A fan blade slows from 30 rad/s to rest with $\alpha = -6.0$ rad/s$^2$: it takes $t = 5.0$ s and turns through $\Delta\theta = \dfrac{0 - 30^2}{2(-6.0)} = 75$ rad –about 12 revolutions.

## Connecting Linear and Rotational Motion

A point at radius $r$ from the axis moves along its arc with

$$s = r\theta, \qquad v = r\omega, \qquad a_t = r\alpha,$$

so points farther out move faster. A rotating point generally has *two* acceleration components at once: the **tangential acceleration** 切向加速度 $a_t = r\alpha$ (speeding up along the arc) and the **centripetal acceleration** 向心加速度 $a_c = \dfrac{v^2}{r} = r\omega^2$ (turning, towards the axis). They are perpendicular, so $a = \sqrt{a_t^2 + a_c^2}$.




*As the radius turns through an angle, a point moves along an arc at speed $v$*

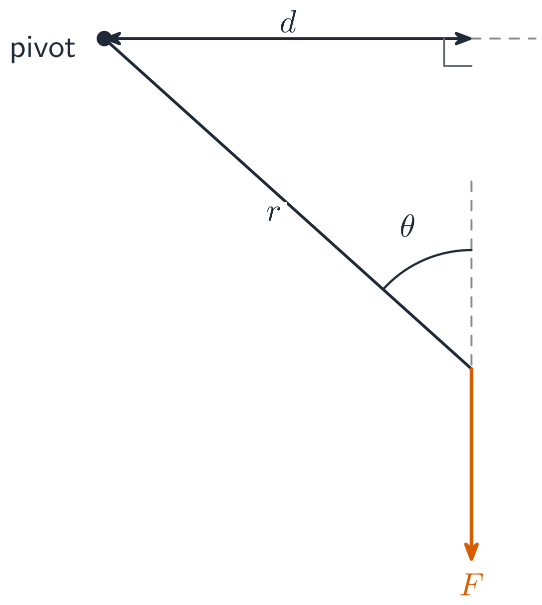
## Torque

**Torque** 力矩 is the turning effect of a force –the rotational cause, as force is the translational cause. It is a **vector product** 矢量积:

$$\vec{\tau} = \vec{r} \times \vec{F}, \qquad \tau = rF\sin\theta = F\,r_\perp,$$

where $r_\perp$, the **moment arm** 力臂, is the perpendicular distance from the axis to the force's line of action. More force, applied farther from the axis, more perpendicular: more torque. A force pointing straight at (or away from) the axis has zero torque. In a plane, give counterclockwise and clockwise torques opposite signs and add.

As a vector product, $\vec{\tau} = \vec{r} \times \vec{F}$ points **perpendicular to the plane** of $\vec{r}$ and $\vec{F}$, with its direction fixed by the **right-hand rule** 右手定则: curl your right fingers from $\vec{r}$ toward $\vec{F}$ and your thumb points along $\vec{\tau}$ –out of the page for a counterclockwise turn, into it for a clockwise one. The same rule gives the direction of angular momentum $\vec{L} = \vec{r} \times \vec{p}$.




*The torque of a force depends on the perpendicular distance from the axis*

# Rotational Inertia

**Rotational inertia 转动惯量** $I$ measures resistance to angular acceleration –the rotational analog of mass, but it depends on *where* the mass sits. For point masses, $I = \sum m_i r_i^2$; for a continuous body,

$$I = \int r^2 \, dm.$$

Mass far from the axis dominates, because of the $r^2$.

**Worked example (rod).** A uniform rod (mass $M$, length $L$) about its centre: with **linear density 线密度** $\lambda = M/L$, $dm = \lambda \, dx$, so $I = \displaystyle\int_{-L/2}^{L/2} x^2 \frac{M}{L} \, dx = \frac{1}{12} M L^2$. AP also expects *nonuniform* rods: if $\lambda(x) = cx$ on $[0, L]$, first find $M = \int_0^L cx \, dx = \frac{1}{2} c L^2$, then $I = \int_0^L x^2 (cx) \, dx = \frac{1}{4} c L^4 = \frac{1}{2} M L^2$ about the light end.

**Worked example (disk from rings).** A solid disk is a nest of rings: a ring at radius $r$ of width $dr$ has $dm = \dfrac{M}{\pi R^2} 2\pi r \, dr$, so

$$I = \int_0^R r^2 \, dm = \frac{2M}{R^2} \int_0^R r^3 \, dr = \tfrac{1}{2} M R^2.$$

The same coaxial-shell method handles cylindrical shells and annular rings.

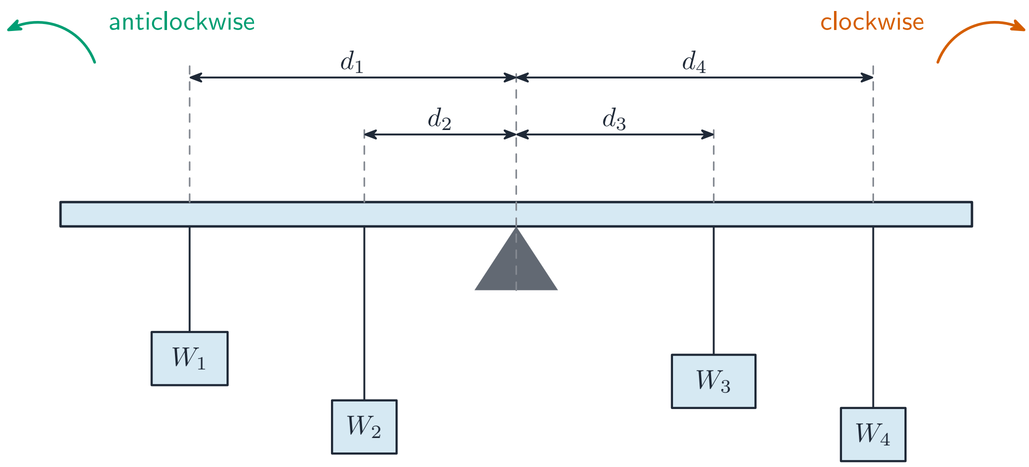
The **parallel-axis theorem 平行轴定理** shifts any known $I$ to a parallel axis a distance $d$ away:

$$I' = I_{\text{cm}} + Md^2.$$

For the rod about one end: $I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$.

## Rotational Equilibrium

Newton's first law has a rotational form: with zero net torque, angular velocity is constant –an object in **rotational equilibrium** 转动平衡 either does not rotate or rotates steadily. Full **static equilibrium** 静平衡 needs both $\sum F = 0$ and $\sum \tau = 0$ (about *any* axis). The standard move for beams and ladders: put the axis at the point where an unknown force acts, so that force drops out of the torque equation.




*At balance the clockwise and anticlockwise torques about the axis are equal*

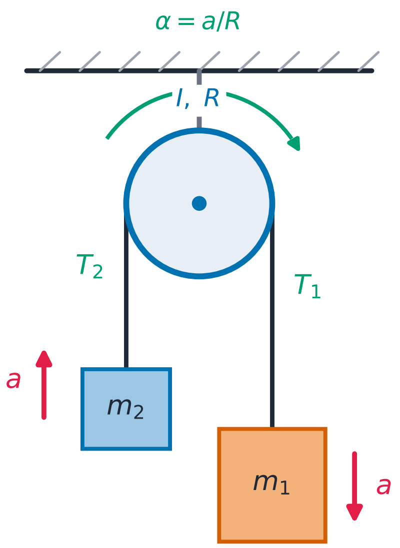
**Worked example.** A uniform 20 kg beam, 4.0 m long, is pivoted at its left end and held horizontal by a vertical rope at its right end. A 40 kg child sits 1.0 m from the pivot. Torques about the pivot: $T(4.0) = 20g(2.0) + 40g(1.0)$, so $T = \dfrac{392 + 392}{4.0} = 196$ N. Then vertical force balance gives the pivot's upward push: $F_p = (60)(9.8) - 196 = 392$ N. Choosing the pivot as the axis removed $F_p$ from the torque equation entirely.

## Newton's Second Law in Rotational Form

A net torque produces angular acceleration in proportion to the rotational inertia:

$$\alpha = \frac{\sum \tau}{I} \qquad \left(\text{more generally } \sum \vec{\tau} = \frac{d\vec{L}}{dt}\right).$$

Solve rotational dynamics exactly like translational dynamics, with $\tau \leftrightarrow F$, $I \leftrightarrow m$, $\alpha \leftrightarrow a$ –free-body diagram, equations, solve.




*A massive pulley: the two tensions differ because the pulley needs net torque*

**Worked example (massive pulley).** Masses $m_1 = 4.0$ kg and $m_2 = 2.0$ kg hang from a rope over a **pulley** 滑轮 of rotational inertia $I = 0.50\ \mathrm{kg \cdot m^2}$ and radius $R = 0.20$ m. Three Newton's-law equations, one per body:

$$m_1 g - T_1 = m_1 a, \qquad T_2 - m_2 g = m_2 a, \qquad (T_1 - T_2)R = I\alpha = \frac{Ia}{R}.$$

Adding them (with $I/R^2 = 12.5$ kg): $a = \dfrac{(m_1 - m_2)g}{m_1 + m_2 + I/R^2} = \dfrac{19.6}{18.5} = 1.1\ \mathrm{m/s^2}$. Then $T_1 = m_1(g - a) = 35$ N and $T_2 = m_2(g + a) = 22$ N. The tensions *must* differ –otherwise nothing would spin the pulley. (If a problem says the pulley is light, then $I \approx 0$ and the tensions become equal again.)

**Exam skill.** Rotational FRQs score the setup: separate free-body diagrams for each mass *and* the pulley, the string constraint $a = R\alpha$ stated, and the sign convention consistent. The single most common error is assuming one tension throughout a rope that passes over a *massive* pulley.

# Exam tips

- Use the rotational analogues: $\theta, \omega = \frac{d\theta}{dt}, \alpha = \frac{d\omega}{dt}$, with the constant-$\alpha$ equations mirroring linear kinematics.
- Convert between linear and angular with $v = r\omega$ and $a_t = r\alpha$ (plus centripetal $a_c = \frac{v^2}{r} = \omega^2 r$).
- Keep radians throughout and fix a positive sense of rotation.
- Relate torque to angular acceleration through $\tau = I\alpha$.
- Draw the rotation axis —the moment arm is the perpendicular distance to it.