# Algebra and graphs

IGCSE Mathematics

This handout covers Topic 2, Algebra and graphs. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

## Working with algebra

In **algebra** 代数 we use letters to stand for numbers. A letter whose value can change is a **variable** 变量. To **substitute** 代入 means to put a number in place of a letter.

**Worked example.** Find the value of $3x^2 - 2y$ when $x = 4$ and $y = 5$.

$$3 \times 4^2 - 2 \times 5 = 3 \times 16 - 10 = 48 - 10 = 38.$$

## Simplifying and expanding

A **term** 项 is a single part of an **expression** 表达式, such as $5a$ or $-9b$. **Like terms** 同类项 have exactly the same letters; you may add or subtract them. The number in front of the letter is the **coefficient** 系数.

**Worked example.** Simplify $2a^2 + 3ab - 1 + 5a^2 - 9ab + 4$.

Collect like terms: $2a^2 + 5a^2 = 7a^2$, $3ab - 9ab = -6ab$, $-1 + 4 = 3$. So the answer is

$$7a^2 - 6ab + 3.$$

To **expand** 展开 means to multiply out **brackets** 括号. Multiply every term inside by the term outside; for two brackets, multiply every term in the first by every term in the second.

**Worked examples.**

$$3x(2x - 4y) = 6x^2 - 12xy.$$

$$(2x + 1)(x - 4) = 2x^2 - 8x + x - 4 = 2x^2 - 7x - 4.$$

For three brackets (**Extended**), expand two first, then multiply by the third:

$$(x - 2)(x + 3)(2x + 1) = (x^2 + x - 6)(2x + 1) = 2x^3 + 3x^2 - 11x - 6.$$

# Factorising

To **factorise** 因式分解 is the opposite of expanding: write the expression as a product of brackets. Always take out the **common factor** 公因式 first.

**Worked example.** $9x^2 + 15xy = 3x(3x + 5y)$, because $3x$ divides both terms.

The following patterns are **Extended**.

**Grouping** (four terms): take a common factor from each pair.

$$xy + 2x + 3y + 6 = x(y + 2) + 3(y + 2) = (x + 3)(y + 2).$$

**Difference of two squares** 平方差: $a^2 - b^2 = (a + b)(a - b)$.

$$9x^2 - 16 = (3x + 4)(3x - 4).$$

**Perfect square** 完全平方: $a^2 + 2ab + b^2 = (a + b)^2$.

$$x^2 + 6x + 9 = (x + 3)^2.$$

**Quadratic** 二次 expressions $ax^2 + bx + c$: find two numbers that multiply to $a \times c$ and add to $b$, then split the middle term.

**Worked example.** Factorise $2x^2 + 7x + 3$.

Here $a \times c = 6$ and $b = 7$. The numbers 1 and 6 work. Split and group:

$$2x^2 + x + 6x + 3 = x(2x + 1) + 3(2x + 1) = (x + 3)(2x + 1).$$

For $ax^3 + bx^2 + cx$, take out the common $x$ first: $2x^3 + 7x^2 + 3x = x(2x^2 + 7x + 3) = x(x + 3)(2x + 1)$.

# Completing the square (Extended)

*A suspension bridge cable hangs in a parabola —the graph of a quadratic.*

**Completing the square** 配方法 rewrites $x^2 + bx + c$ as $(x + p)^2 + q$. Take half of the $x$-coefficient, square it, then balance.

**Worked example.** Write $x^2 + 6x + 1$ in completed-square form.

Half of 6 is 3, and $3^2 = 9$:

$$x^2 + 6x + 1 = (x + 3)^2 - 9 + 1 = (x + 3)^2 - 8.$$

When the $x^2$ has a coefficient, take it out of the first two terms first:

$$2x^2 + 8x + 3 = 2(x^2 + 4x) + 3 = 2\big((x + 2)^2 - 4\big) + 3 = 2(x + 2)^2 - 5.$$

# Algebraic fractions (Extended)

An **algebraic fraction** 分式 has algebra on the top or bottom. Add and subtract using a common denominator; multiply and divide as with ordinary fractions.

**Worked examples.**

$$\frac{x}{3} + \frac{x - 4}{2} = \frac{2x}{6} + \frac{3(x - 4)}{6} = \frac{2x + 3x - 12}{6} = \frac{5x - 12}{6}.$$

$$\frac{3a}{4} \div \frac{9a}{10} = \frac{3a}{4} \times \frac{10}{9a} = \frac{30a}{36a} = \frac{5}{6}.$$

To simplify a **rational expression** 有理式, factorise the top and bottom, then cancel common brackets.

$$\frac{x^2 - 2x}{x^2 - 5x + 6} = \frac{x(x - 2)}{(x - 2)(x - 3)} = \frac{x}{x - 3}.$$

# Indices in algebra

The laws of **indices** 指数 work with letters too: $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, and $(a^m)^n = a^{mn}$.

**Worked examples.**

$$(5x^3)^2 = 25x^6, \qquad 12a^5 \div 3a^{-2} = 4a^7, \qquad 6x^7y^4 \times 5x^{-5}y = 30x^2y^5.$$

You can also solve simple index equations by writing both sides with the same **base** 底数.

**Worked example.** Solve $2^x = 32$. Since $32 = 2^5$, you get $x = 5$.

# Equations

An **equation** 方程 says two expressions are equal. To solve a **linear** 一次 equation, do the same operation to both sides until the **unknown** 未知数 is alone.

**Worked example.** Solve $5 - 2x = 3(x + 7)$.

$$5 - 2x = 3x + 21 \;\Rightarrow\; 5 - 21 = 3x + 2x \;\Rightarrow\; -16 = 5x \;\Rightarrow\; x = -\tfrac{16}{5}.$$

## Fractional equations (Extended)

A **fractional equation** 分式方程 has the unknown in a denominator. Multiply both sides by the denominator to clear it.

**Worked example.** Solve $\dfrac{x}{2x + 1} = 4$.

$$x = 4(2x + 1) = 8x + 4 \;\Rightarrow\; -7x = 4 \;\Rightarrow\; x = -\tfrac{4}{7}.$$

## Simultaneous equations

**Simultaneous equations** 联立方程 are two equations solved together. For two linear equations, add or subtract to remove one letter.

**Worked example.** Solve $2x + y = 7$ and $3x - y = 8$.

Adding removes $y$: $5x = 15$, so $x = 3$. Then $y = 7 - 2(3) = 1$.

For one linear and one **quadratic** equation (**Extended**), substitute the linear into the curve.

**Worked example.** Solve $y = x + 2$ and $y = x^2$.

$$x^2 = x + 2 \;\Rightarrow\; x^2 - x - 2 = 0 \;\Rightarrow\; (x - 2)(x + 1) = 0,$$

so $x = 2$ (giving $y = 4$) or $x = -1$ (giving $y = 1$).

## Solving quadratic equations (Extended)

There are three methods.

- **By factorising:** $x^2 + 5x + 6 = 0 \Rightarrow (x + 2)(x + 3) = 0 \Rightarrow x = -2$ or $x = -3$.
- **By completing the square:** $x^2 + 6x + 1 = 0 \Rightarrow (x + 3)^2 = 8 \Rightarrow x + 3 = \pm 2\sqrt{2} \Rightarrow x = -3 \pm 2\sqrt{2}$.
- **By the quadratic formula** 求根公式, which is given in the exam:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

**Worked example (formula).** Solve $2x^2 + 3x - 1 = 0$. Here $a = 2$, $b = 3$, $c = -1$:

$$x = \frac{-3 \pm \sqrt{9 + 8}}{4} = \frac{-3 \pm \sqrt{17}}{4}.$$

## Changing the subject

To **change the subject** 公式变形 of a formula means to rearrange it so a chosen letter is alone on one side.

**Worked example.** Make $r$ the subject of $A = \pi r^2$ (**Extended**, because of the power).

$$r^2 = \frac{A}{\pi} \;\Rightarrow\; r = \sqrt{\frac{A}{\pi}}.$$

When the letter appears twice (**Extended**), collect those terms and factorise. To make $x$ the subject of $y = \dfrac{x+1}{x-1}$:

$$y(x-1) = x+1 \;\Rightarrow\; yx - x = 1 + y \;\Rightarrow\; x(y-1) = 1 + y \;\Rightarrow\; x = \frac{1+y}{y-1}.$$

# Inequalities

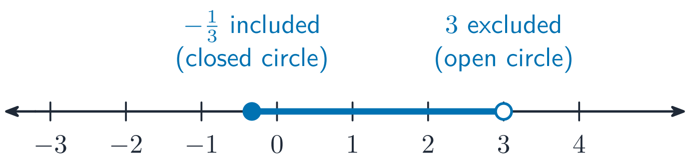
An **inequality** 不等式 uses $<$, $>$, $\leqslant$ or $\geqslant$. Solve it like an equation, but **reverse the sign** if you multiply or divide by a negative number.

**Worked example.** Solve $-3 \leqslant 3x - 2 < 7$.

Add 2 to all parts, then divide by 3:

$$-1 \leqslant 3x < 9 \;\Rightarrow\; -\tfrac{1}{3} \leqslant x < 3.$$

On a **number line** 数轴, use an open circle for $<$ or $>$ (value not included) and a closed circle for $\leqslant$ or $\geqslant$ (value included).




$-\frac{1}{3} \leqslant x < 3$: *a closed circle includes the end value, an open circle excludes it.*

## Regions (Extended)

An inequality in two letters describes a **region** 区域 of the graph. Draw the boundary line (broken for $<$ or $>$, solid for $\leqslant$ or $\geqslant$) and shade the unwanted side. You may also be asked to list the inequalities that define a given region.

## Sequences

*Romanesco broccoli: self-similar spirals form a natural number pattern.*

Image: Jon Sullivan, Public domain (commons.wikimedia.org)

A **sequence** 数列 is a list of numbers that follow a rule. The **term-to-term rule** 递推规则 tells you how to get the next term from the one before.

To find the rule for the term in position $n$ (the $n$th **term**), look at how the terms change.

**Linear sequence** (the terms go up by the same amount each time). The difference is the multiple of $n$.

**Worked example.** Find the $n$th term of $2, 5, 8, 11, \dots$

The terms go up by 3, so start with $3n$. Since $3 \times 1 = 3$ but the first term is 2, subtract 1: the $n$th term is $3n - 1$.

**Quadratic sequence** (the differences themselves change by the same amount). The second difference equals $2\times$ the coefficient of $n^2$.

**Worked example.** Find the $n$th term of $2, 5, 10, 17, \dots$

First differences are $3, 5, 7$; the second difference is 2, so the $n^2$ part is $1n^2$. Subtracting $n^2$ $(1, 4, 9, 16)$ from the sequence leaves $1, 1, 1, 1$. So the $n$th term is $n^2 + 1$.

A **cubic** 三次 sequence such as $1, 8, 27, 64, \dots$ has $n$th term $n^3$. An **exponential sequence** 指数数列 such as $2, 6, 18, 54, \dots$ multiplies by a fixed number each time; here the $n$th term is $2 \times 3^{n-1}$.

# Direct and inverse proportion (Extended)

Two quantities are in **direct proportion** 正比例 if one is always a fixed multiple of the other: $y \propto x$ means $y = kx$, where $k$ is a **constant** 常数. They are in **inverse proportion** 反比例 if one rises as the other falls: $y \propto \dfrac{1}{x}$ means $y = \dfrac{k}{x}$. The symbol $\propto$

is read "is proportional to". You can also have proportion to a square, square root, cube or cube root.

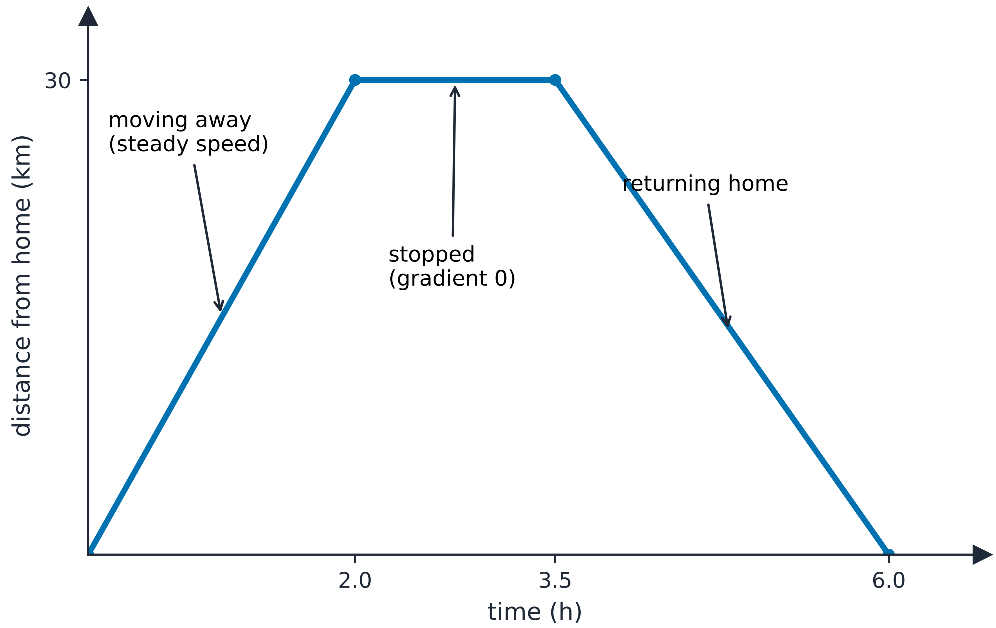
**Worked example.** $y$ is in direct proportion to $x$, and $y = 12$ when $x = 3$. Find $y$ when $x = 7$.

First find $k$: $12 = k \times 3$, so $k = 4$ and $y = 4x$. Then $y = 4 \times 7 = 28$.

# Graphs in practical situations

The **gradient** 斜率 (steepness) of a graph shows a **rate of change** 变化率.

- A **travel graph** 行程图 (distance–time graph) has gradient equal to speed; a flat part means the object is not moving.
- A **conversion graph** 换算图 is a straight line used to change between two units (for example, miles and kilometres).




*On a distance–time graph the gradient is the speed; a flat section means the object has stopped.*

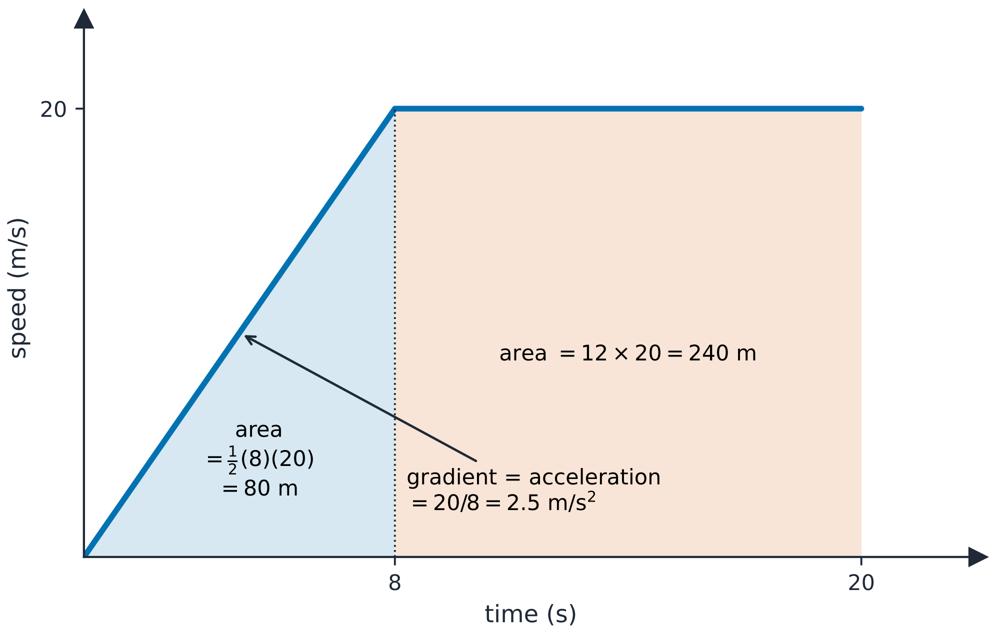
## Speed–time graphs (Extended)

On a speed–time graph the gradient is the **acceleration** 加速度 (or **deceleration** 减速度 if the speed falls), and the **area** 面积 under the graph is the **distance** 距离 travelled.

**Worked example.** A car speeds up from rest to 20 m/s in 8 s, then stays at 20 m/s for 12 s. Find the acceleration and the total distance.

Acceleration $= \dfrac{20}{8} = 2.5\ \text{m/s}^2$. The distance is the area: a triangle plus a rectangle,

$$\tfrac{1}{2} \times 8 \times 20 + 12 \times 20 = 80 + 240 = 320\ \text{m}.$$




*On a speed–time graph the gradient is the acceleration and the area underneath is the distance travelled.*

## Graphs of functions

To draw a graph, make a **table of values** 数值表: choose values of $x$, work out $y$, then plot the **coordinates** 坐标 and join them with a smooth curve.

The points where a graph crosses the $x$-axis (the horizontal **axis** 坐标轴) are the **roots** 根—the solutions of $y = 0$.

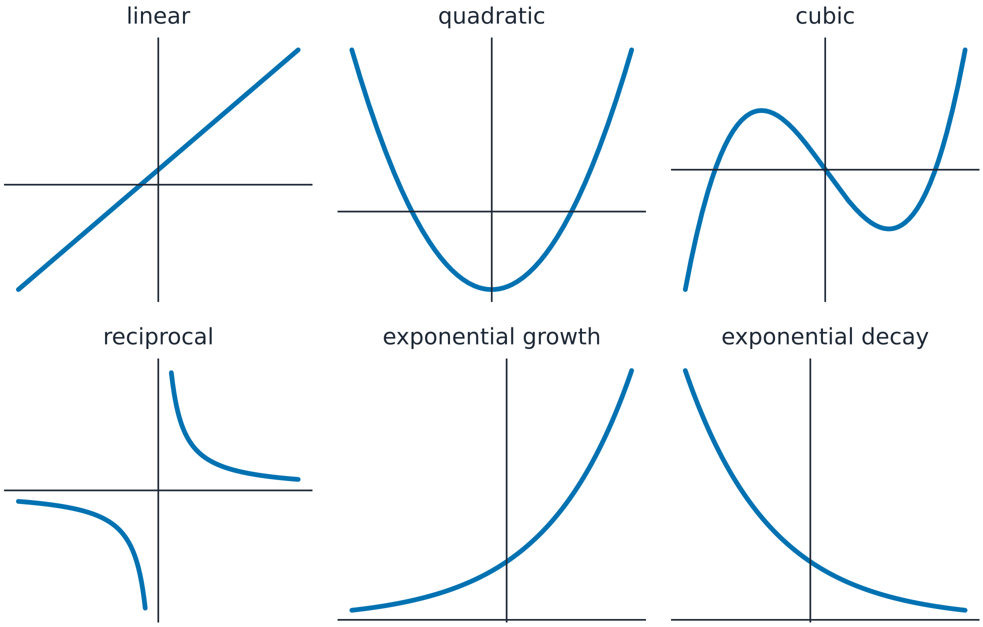
You can solve an equation by reading a graph. The **intersection point** 交点 of a line and a curve gives the solution of the two equations together.

For **exponential growth** 指数增长 and **exponential decay** 指数衰减, the graph of $y = a\,b^x + c$ rises (or falls) faster and faster and flattens towards a horizontal line.

## Sketching curves

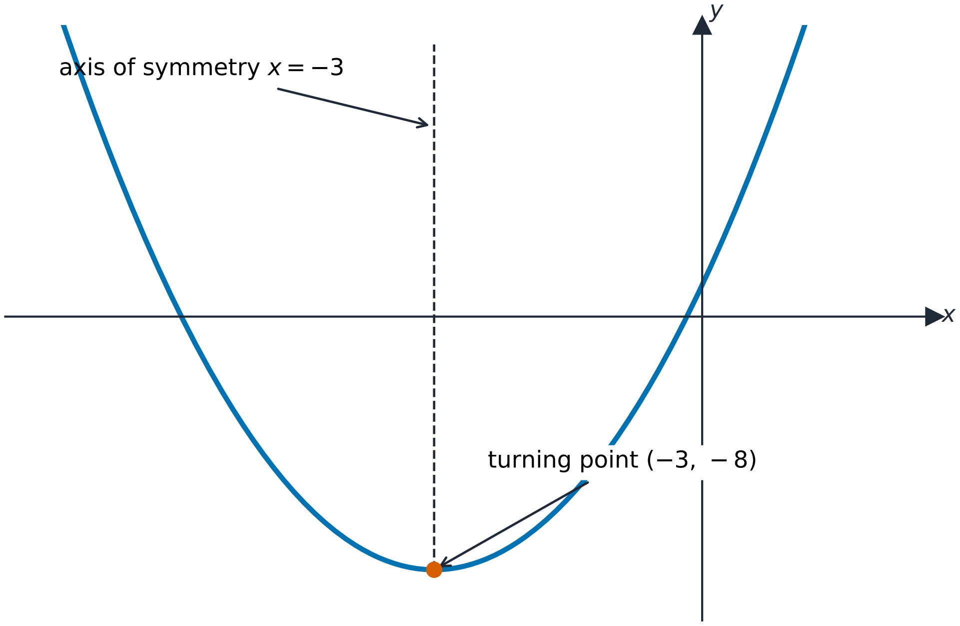
A quick sketch should show the right shape and the key features: where it crosses the axes, any **symmetry** 对称, and any line the curve gets close to.

| **Function** | **Shape** |
|---|---|
| linear, $y = mx + c$ | straight line; gradient $m$, $y$-**intercept** 截距 $c$ |
| quadratic, $y = ax^2 + bx + c$ | a **parabola** 抛物线 (U-shape if $a > 0$, ∩-shape if $a < 0$) |
| cubic, $y = ax^3 + bx + c$ | an S-shaped curve |
| reciprocal, $y = \dfrac{a}{x} + b$ | two separate curves |
| exponential, $y = a\,r^x + b$ | fast growth or decay |




*The basic graph shapes; knowing each shape lets you sketch quickly from the equation.*

For a parabola, completing the square gives the **turning point** 转折点 (the lowest or highest point). For example $y = (x + 3)^2 - 8$ has its turning point at $(-3, -8)$.

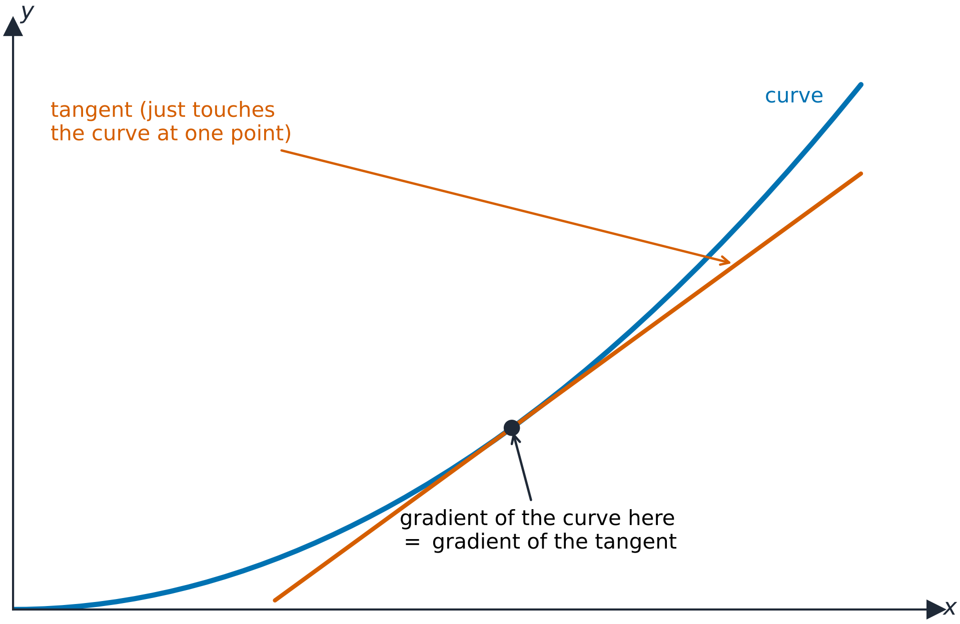



*Completing the square, $y = (x + 3)^2 - 8$, shows the turning point $(-3, -8)$ and the line of symmetry $x = -3$.*

An **asymptote** 渐近线 is a line that the curve gets closer and closer to but never touches —for example, the $x$-axis for $y = \dfrac{a}{x}$, or the line $y = b$ for $y = a\,r^x + b$.

# Differentiation (Extended)

**Differentiation** 微分 finds the gradient of a curve at any point. You can estimate it by drawing a **tangent** 切线 (a line that just touches the curve) and measuring its gradient.




*The gradient of a curve at a point equals the gradient of the tangent there —what differentiation finds.*

The exact rule: if $y = ax^n$, then the **derivative** 导数 is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\,n\,x^{n-1}.$$

Differentiate a sum term by term.

**Worked example.** If $y = x^3 + 2x^2 - 5x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 4x - 5$.

A **stationary point** 驻点 (turning point) is where the gradient is zero, so set $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

**Worked example.** Find the turning point of $y = x^2 - 6x + 5$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 6 = 0 \;\Rightarrow\; x = 3, \quad y = 3^2 - 6(3) + 5 = -4.$$

The turning point is $(3, -4)$. To decide whether a turning point is a **maximum** 最大值 or a **minimum** 最小值, check the sign of the gradient on each side, or use the second derivative (positive means a minimum).

# Functions (Extended)

A **function** 函数 turns each input into one output. We write $f(x)$, for example $f(x) = 3x - 5$, so $f(2) = 1$. The set of allowed inputs is the **domain** 定义域; the set of possible outputs is the **range** 值域.

The **inverse function** 反函数 $f^{-1}(x)$ undoes the function. To find it, write $y = f(x)$, swap the roles, and make $x$ the subject.

**Worked example.** Find the inverse of $f(x) = 3x - 5$.

$$y = 3x - 5 \;\Rightarrow\; x = \frac{y+5}{3}, \quad \text{so} \quad f^{-1}(x) = \frac{x+5}{3}.$$

A **composite function** 复合函数 applies one function after another: $gf(x)$ means "do $f$ first, then $g$".

**Worked example.** If $f(x) = 2x$ and $g(x) = x + 3$, then

$$gf(x) = g(2x) = 2x + 3, \qquad fg(x) = f(x+3) = 2(x+3) = 2x + 6.$$