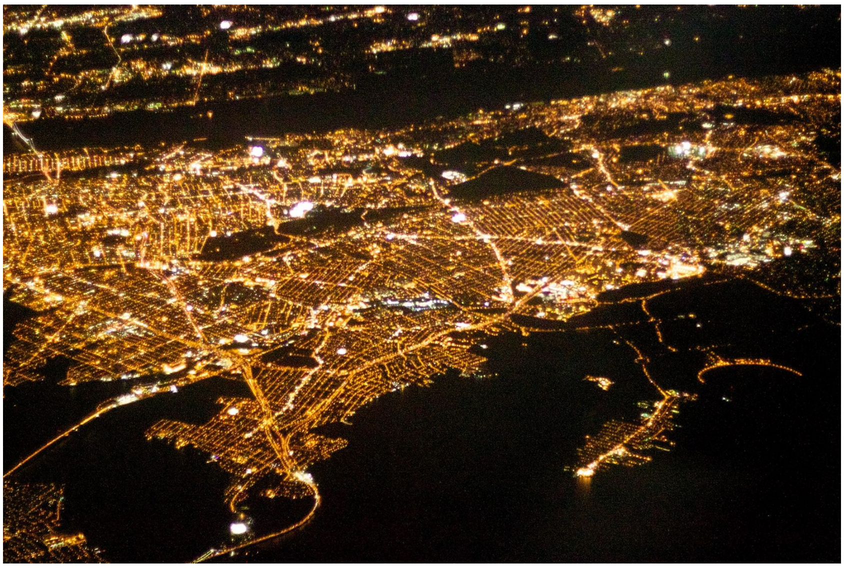
Coordinate geometry

IGCSE Mathematics

This handout covers Topic 3, Coordinate geometry. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Coordinates

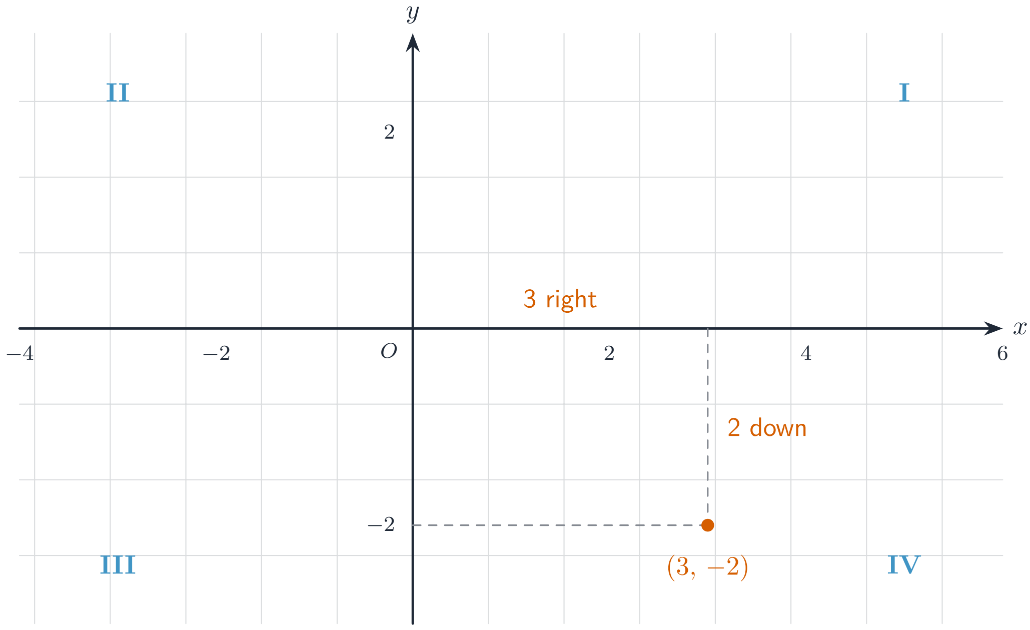


A city street grid: every place is fixed by its coordinates.

A point on a graph is described by its **coordinates** 坐标, sometimes called Cartesian coordinates, written (x, y). The first number is the across value and the second is the up value.

- The two number lines are the **axes** 坐标轴: the **horizontal** 水平 x-axis and the **vertical** 竖直 y-axis.
- They cross at the **origin** 原点, the point $(0, 0)$.
- The axes split the grid into four **quadrants** 象限.

So the point $(3, -2)$ is found by going 3 to the right and 2 down.



The axes meet at the origin O and split the plane into four quadrants; $(3, -2)$ means 3 right then 2 down.

The equation of a straight line

Most straight lines can be written as

$$y = mx + c,$$

where m is the **gradient** 斜率 (steepness) and c is the **intercept** 截距—the y-value where the line crosses the y-axis.

- A line like $x = k$ (for example $x = 3$) is **vertical**.
- A line like $y = k$ (for example $y = 3$) is **horizontal**.

A line may also be given as $ax + by = c$. Rearrange it into $y = mx + c$ to read off the gradient and intercept.

Worked example. Find the gradient and y-intercept of $5x + 4y = 8$.

$$4y = -5x + 8 \;\Rightarrow\; y = -\tfrac{5}{4}x + 2.$$

So the gradient is $-\frac{5}{4}$ and the y-intercept is 2.

Gradient

A steep mountain road: gradient measures steepness as rise over run.

Image: Philipp Ramseier, CC BY-SA 4.0 (commons.wikimedia.org)

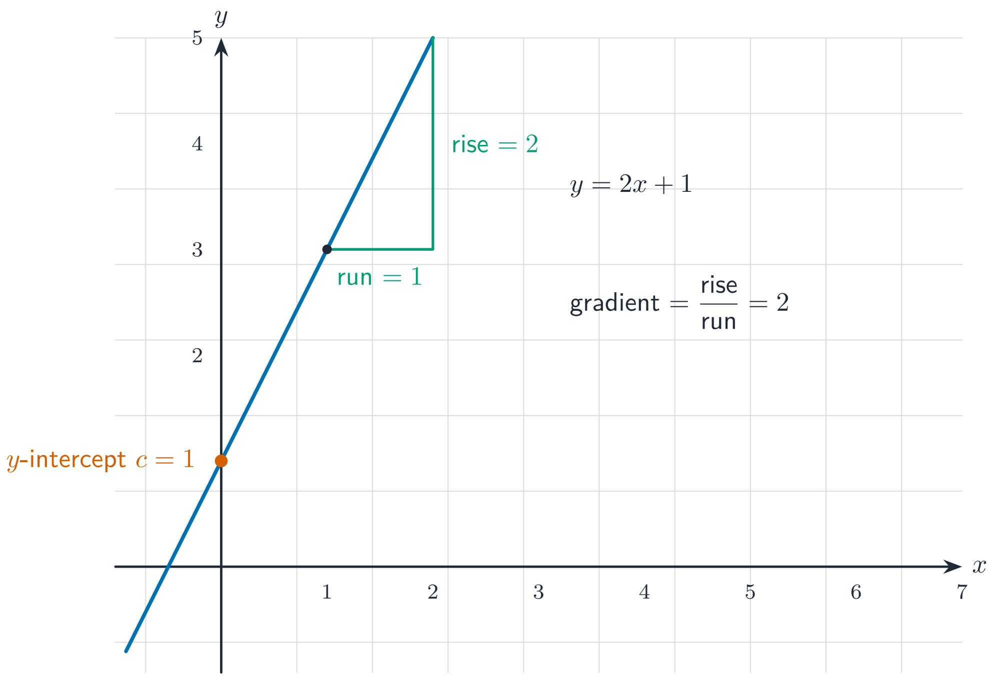
The **gradient** measures how steep a line is:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}.$$

A positive gradient goes up to the right; a negative gradient goes down to the right.

Worked example. Find the gradient of the line through $(1, 2)$ and $(4, 11)$.

$$m = \frac{11 - 2}{4 - 1} = \frac{9}{3} = 3.$$



For $y = mx + c$, the line crosses the y-axis at c and the gradient m is the rise divided by the run.

Drawing a straight-line graph

To draw $y = mx + c$, the quickest way is:

1. Mark the intercept c on the y-axis.
2. Use the gradient to step to more points (for $m = 3$, go 1 right and 3 up).
3. Join the points with a straight line.

You can also make a small **table of values 数值表**: choose two or three x-values, work out y, and plot the points.

Finding the equation of a line

If you know the gradient m and one point on the line, put the point into $y = mx + c$ to find c.

Worked example. A line has gradient 3 and passes through $(1, 2)$. Find its equation.

$$y = 3x + c, \qquad 2 = 3(1) + c, \qquad c = -1.$$

So the equation is $y = 3x - 1$. (If you are given two points, first find the gradient, then do this.)

Length and midpoint (Extended)

A **line segment 线段** is the straight piece between two points.

To find the **length** 长度 of the segment between (x_1, y_1) and (x_2, y_2), use **Pythagoras' theorem** 勾股定理 on the horizontal and vertical gaps:

$$\text{length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

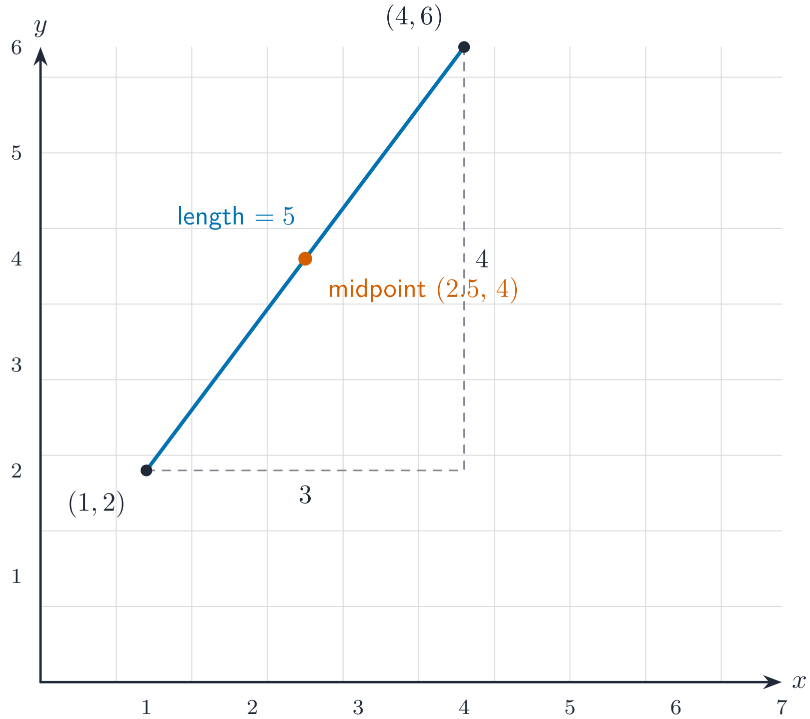
To find the **midpoint** 中点 (the point exactly in the middle), average the coordinates:

$$\text{midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Worked example. Find the length and midpoint of the segment from $(1, 2)$ to $(4, 6)$.

$$\text{length} = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

$$\text{midpoint} = \left(\frac{1+4}{2}, \frac{2+6}{2}\right) = (2.5,\ 4).$$



The horizontal gap (3) and vertical gap (4) make a right triangle, so the length is $\sqrt{3^2 + 4^2} = 5$*; the midpoint is the average of the coordinates.*

Parallel lines

Parallel 平行 lines never meet, so they have the **same gradient**.

Worked example. Find the equation of the line parallel to $y = 4x - 1$ that passes through $(1, -3)$.

The gradient is also 4. Put the point in:

$$-3 = 4(1) + c \;\Rightarrow\; c = -7,$$

so the line is $y = 4x - 7$.

Perpendicular lines (Extended)

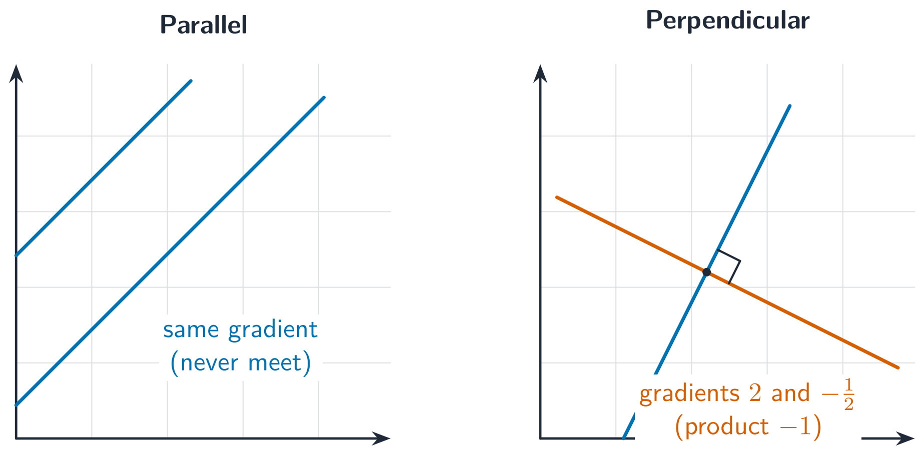
Two lines are **perpendicular** 垂直 if they meet at a **right angle** 直角. Their gradients multiply to -1:

$$m_1 \times m_2 = -1, \qquad \text{so} \qquad m_2 = -\frac{1}{m_1}.$$

In words: flip the fraction and change the sign.

Worked example. Find the gradient of a line perpendicular to $2y = 3x + 1$.

Rearrange: $y = \frac{3}{2}x + \frac{1}{2}$, so the gradient is $\frac{3}{2}$. The perpendicular gradient is $-\frac{2}{3}$.



Parallel lines share the same gradient; perpendicular gradients multiply to -1.

Perpendicular bisector

The **perpendicular bisector** 垂直平分线 of a segment cuts it in half at a right angle. To find its equation: get the midpoint, then use the perpendicular gradient through that midpoint.

Worked example. Find the perpendicular bisector of the segment joining $(-3, 8)$ and $(9, -2)$.

- Midpoint: $\left(\frac{-3+9}{2}, \frac{8+(-2)}{2}\right) = (3, 3)$.
- Gradient of the segment: $\frac{-2-8}{9-(-3)} = \frac{-10}{12} = -\frac{5}{6}$.
- Perpendicular gradient: $\frac{6}{5}$.

Through $(3,3)$: $3 = \frac{6}{5}(3) + c \Rightarrow c = 3 - \frac{18}{5} = -\frac{3}{5}$. So the bisector is

$$y = \frac{6}{5}x - \frac{3}{5}.$$