Trigonometry

IGCSE Mathematics

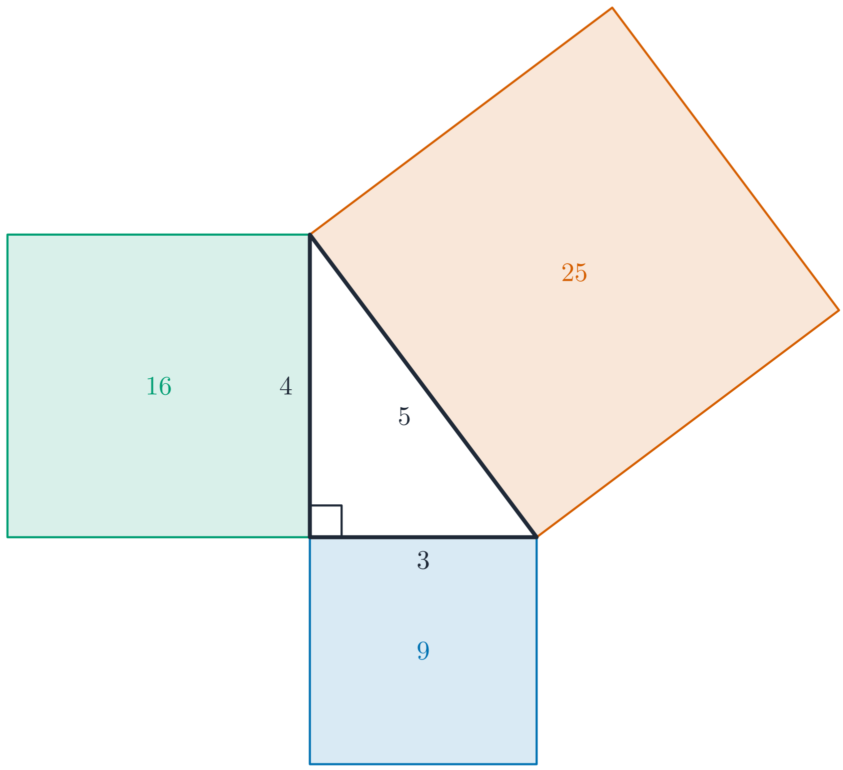
This handout covers Topic 6, Trigonometry. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels. Give angle answers to one decimal place.

Pythagoras' theorem

Pythagoras' theorem 勾股定理 links the three sides of a **right-angled triangle** 直角三角形 (a **triangle** 三角形 with one 90° angle). If the longest side (the **hypotenuse** 斜边, opposite the right angle) is c, then

$$a^2 + b^2 = c^2.$$

Use it to find a missing side.



Pythagoras as areas: the squares on the two shorter sides (9 + 16) add up to the square on the hypotenuse (25).

Worked example. A right-angled triangle has a hypotenuse of 13 cm and one short side of 5 cm. Find the other short side.

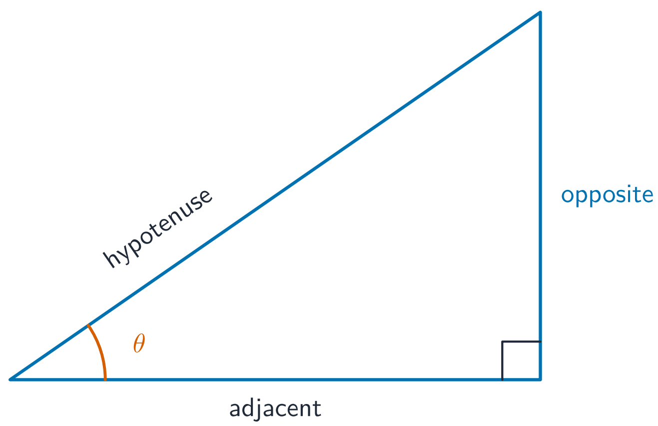
$$b^2 = 13^2 - 5^2 = 169 - 25 = 144, \qquad b = \sqrt{144} = 12 \text{ cm}.$$

Trigonometry in right-angled triangles

Label the sides from the angle you are using: the **opposite** 对边 (across from the angle), the **adjacent** 邻边 (next to the angle), and the hypotenuse. The three ratios are the **sine** 正弦, **cosine** 余弦 and **tangent ratio** 正切 (sin, cos, tan):

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

Remember them as **SOH-CAH-TOA**. To find an angle, use the inverse ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$).



Name the sides from the angle θ: the opposite is across from it, the adjacent next to it, and the hypotenuse opposite the right angle (SOH-CAH-TOA).

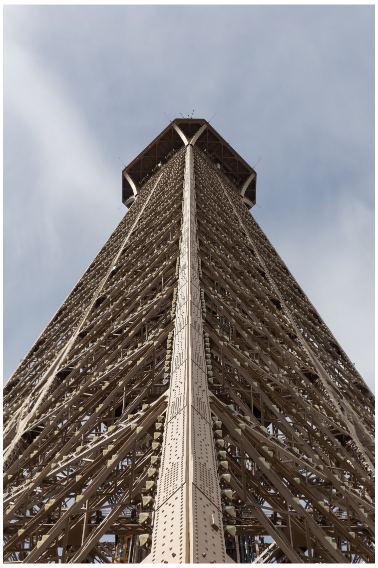
Worked example (find a side). In a right-angled triangle the hypotenuse is 10 cm and the angle is 30°. Find the opposite side.

$$\text{opp} = 10 \times \sin 30^\circ = 10 \times 0.5 = 5 \text{ cm}.$$

Worked example (find an angle). The opposite side is 4 cm and the adjacent side is 3 cm.

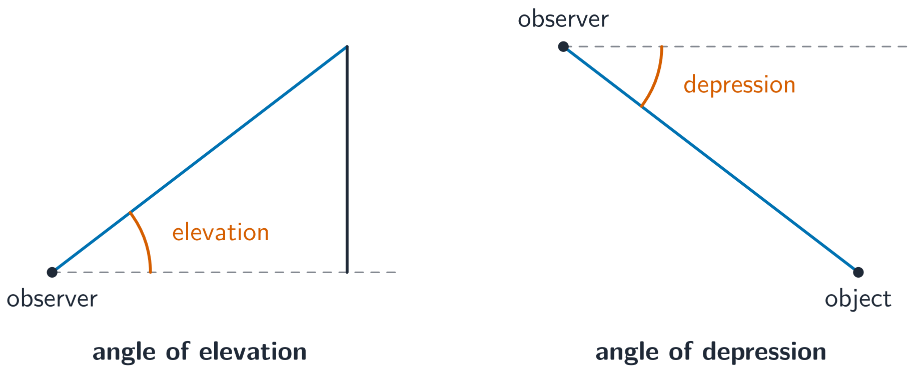
$$\tan\theta = \frac{4}{3}, \qquad \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ.$$

Angles of elevation and depression (Extended)



Looking up at a tower involves an angle of elevation.

The **angle of elevation** 仰角 is the angle up from the horizontal to an object above you. The **angle of depression** 俯角 is the angle down from the horizontal to an object below you. The shortest distance from a point to a line is the **perpendicular** 垂直 distance.



The angle of elevation looks up from the horizontal; the angle of depression looks down.

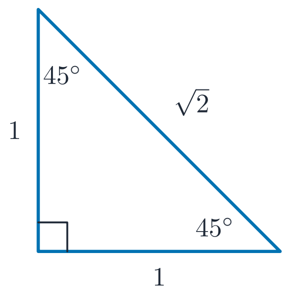
Worked example. From a point 50 m from the foot of a tower, the angle of elevation to the top is 40°. Find the height of the tower.

$$\text{height} = 50 \times \tan 40^\circ = 50 \times 0.839 = 42.0 \text{ m}.$$

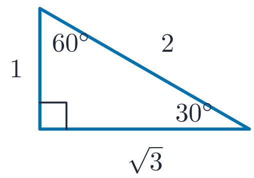
Exact trigonometric values (Extended)

You must know these exact values without a calculator.

x	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	—



45° **triangle**



30°–60° **triangle**

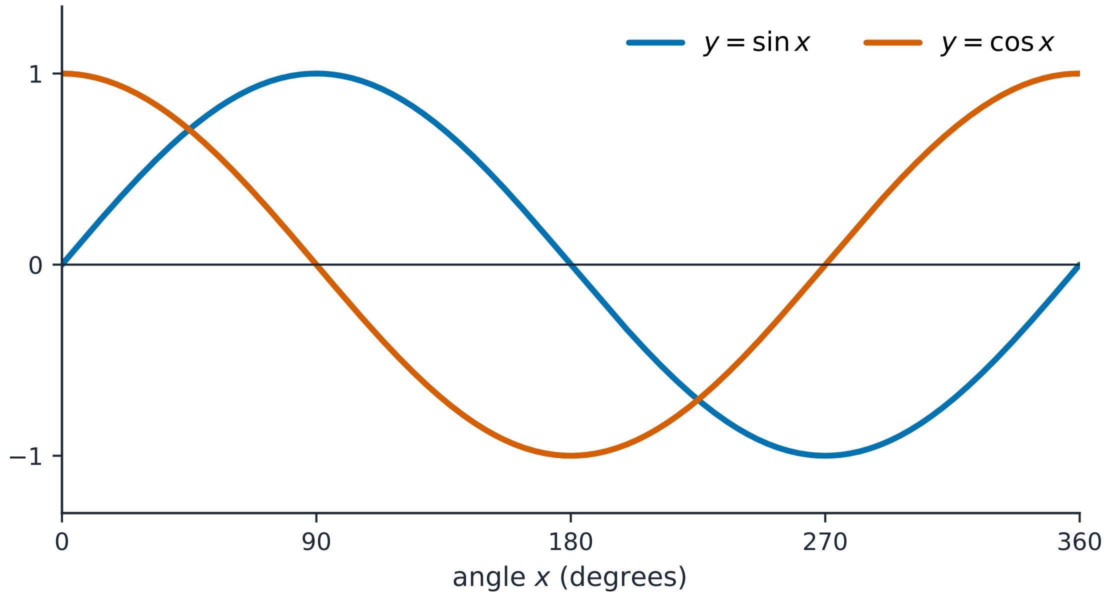
These two special triangles are where the exact values come from —worth memorising.

Graphs and trigonometric equations (Extended)

A Ferris wheel: a point on the rim traces a sine curve as it turns.

For $0° \leqslant x \leqslant 360°$:

- $y = \sin x$ is a wave starting at 0, peaking at 90°, back to 0 at 180°, down to -1 at 270°.
- $y = \cos x$ is the same wave but starting at 1.
- $y = \tan x$ rises steeply and repeats every 180°.



$y = \sin x$ and $y = \cos x$ are smooth waves between -1 and 1; cosine is sine shifted left by $90°$.

A **trigonometric equation** 三角方程 often has more than one answer in this range. Use the graph (or the symmetry of the wave) to find them all.

Worked example. Solve $\sin x = \frac{\sqrt{3}}{2}$ for $0° \leqslant x \leqslant 360°$.

One answer is $x = 60°$. The sine wave is also $\frac{\sqrt{3}}{2}$ at $180° - 60° = 120°$. So $x = 60°$ or $120°$.

Worked example. Solve $2\cos x + 1 = 0$ for $0° \leqslant x \leqslant 360°$.

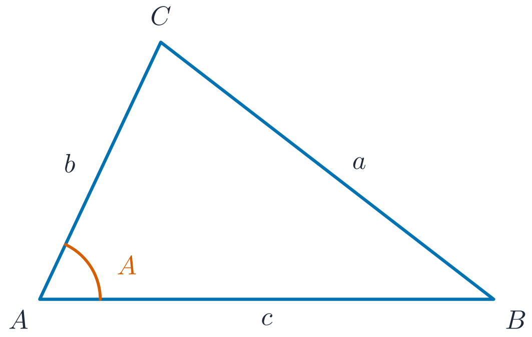
$$\cos x = -\tfrac{1}{2} \;\Rightarrow\; x = 120° \text{ or } 240°.$$

The sine and cosine rules (Extended)

For **any** triangle (not just right-angled), with sides a, b, c opposite angles A, B, C:

$$\text{sine rule:} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

$$\text{cosine rule:} \quad a^2 = b^2 + c^2 - 2bc\cos A.$$



In any triangle, sides a, b, c lie opposite the angles A, B, C of the same letter.

Use the **sine rule** 正弦定理 when you have a side and its opposite angle. Use the **cosine rule** 余弦定理 when you have two sides and the angle between them, or all three sides. With the sine rule, watch for the **ambiguous case** 两解情况, where an angle could be acute or obtuse.

The area of any triangle is

$$\text{area} = \tfrac{1}{2}ab\sin C.$$

Worked example. A triangle has $b = 7$ cm, $c = 8$ cm and the angle between them $A = 40°$. Find side a.

$$a^2 = 7^2 + 8^2 - 2(7)(8)\cos 40° = 113 - 112 \times 0.766 = 27.2,$$

$$a = \sqrt{27.2} = 5.2 \text{ cm}.$$

Pythagoras and trigonometry in 3D (Extended)

In three dimensions, find a right-angled triangle inside the solid, then use Pythagoras or trigonometry on it. A common task is the angle between a line and a flat surface (a **plane** 平面).

Worked example. A box has a base 6 cm by 8 cm and height 5 cm. Find the angle between a space diagonal and the base.

First the base diagonal: $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$ cm. This diagonal and the height form a right-angled triangle, so the angle θ with the base is

$$\tan\theta = \frac{5}{10} = 0.5, \qquad \theta = \tan^{-1}(0.5) = 26.6°.$$