Transformations and vectors

IGCSE Mathematics

This handout covers Topic 7, Transformations and vectors. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels. Vectors as a whole topic are Extended.

Transformations

Geometric tiles are built by reflecting, rotating and translating shapes.

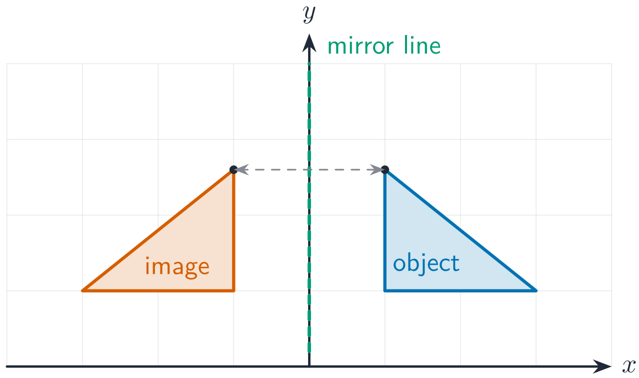
A **transformation** 变换 changes the position or size of a shape. The original is the *object* and the result is the *image*. There are four types. When asked to **describe** one, you must name the type and give all the details below.

Reflection

A **reflection** 反射 flips the shape over a **mirror line** 对称轴. Each image point is the same distance from the line as the object point, on the other side.

- To describe it, give the **equation of the mirror line** (for Core, a horizontal or vertical line; Extended allows any line such as $y = x$).

Worked example. Reflect the point $(3, 2)$ in the y-axis. Only the sign of x changes: the image is $(-3, 2)$. (In the x-axis it would be $(3, -2)$.)



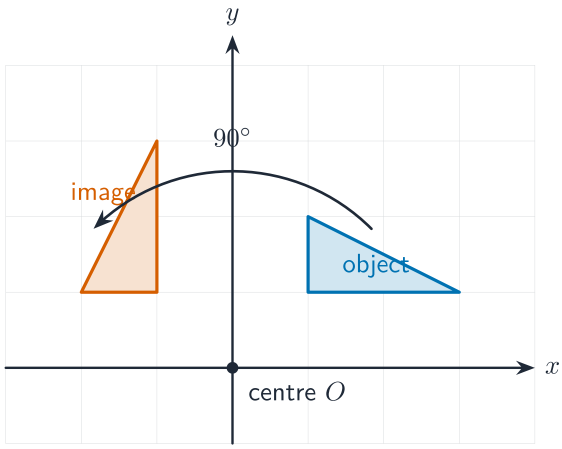
A reflection flips the shape over a mirror line (here the y-axis); each point and its image are the same distance from the line.

Rotation

A **rotation** 旋转 turns the shape about a fixed point, the **centre** 中心 of rotation, through multiples of 90°.

- To describe it, give the centre, the angle, and the direction (clockwise or anticlockwise).

Worked example. Rotate $(3, 1)$ by $90°$ anticlockwise about the origin. The rule is $(x, y) \to (-y, x)$, so the image is $(-1, 3)$.



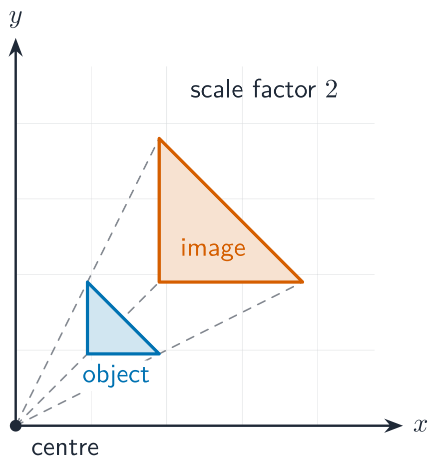
A rotation turns the shape about a fixed centre —here $90°$ *anticlockwise about the origin.*

Enlargement

An **enlargement** 放大 changes the size by a **scale factor** 比例因子 k, measured from a fixed centre. Each distance from the centre is multiplied by k.

- To describe it, give the centre and the scale factor. For Extended, k may be **fractional** (the shape gets smaller) or **negative** (the image appears on the other side of the centre).

Worked example. Enlarge $(1, 2)$ from the origin by scale factor 2. Multiply both coordinates: the image is $(2, 4)$.

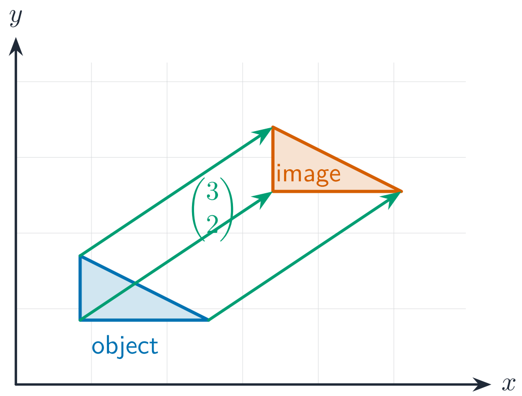


An enlargement multiplies every distance from the centre by the scale factor (here 2).

Translation

A **translation** 平移 slides the shape with no turning, by a **vector** 向量 written as a **column vector** 列向量 $\begin{pmatrix} x \\ y \end{pmatrix}$ (x across, y up).

Worked example. Translate $(5, 3)$ by $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$: move 2 left and 4 up to get $(3, 7)$.



A translation slides every point by the same column vector, with no turning.

(**Extended:** a question may ask you to combine two transformations and describe the single transformation that has the same effect.)

Vectors in two dimensions (Extended)

Forces like wind are vectors, with both size and direction.

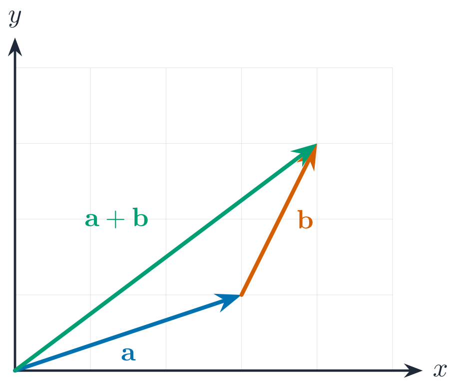
Image: Ermell, CC BY-SA 4.0 (commons.wikimedia.org)

A vector has both size and direction. It can be written as a column vector, as $\overrightarrow{AB}$ (from A to B), or in bold as $\mathbf{a}$.

- **Add or subtract** by working on the top and bottom numbers separately.
- **Multiply by a scalar** 标量 (an ordinary number) by multiplying both numbers.

Worked example. If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$, then

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \qquad 3\mathbf{a} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}.$$



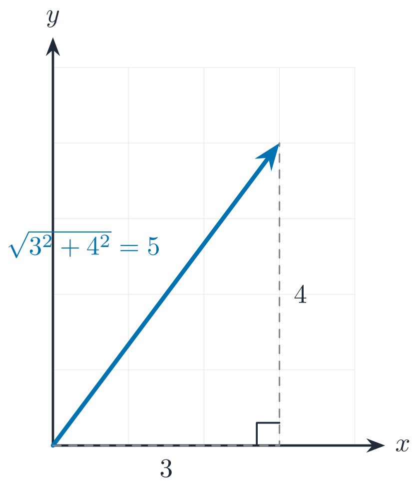
Adding by the triangle law: draw $\mathbf{b}$ *from the tip of* $\mathbf{a}$*, and* $\mathbf{a} + \mathbf{b}$ *runs from start to finish.*

Magnitude of a vector (Extended)

The **magnitude** 模 (length) of a vector is found with Pythagoras. For $\begin{pmatrix} x \\ y \end{pmatrix}$,

$$\left|\begin{pmatrix} x \\ y \end{pmatrix}\right| = \sqrt{x^2 + y^2}.$$

Worked example. The magnitude of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.



The magnitude of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ *comes from Pythagoras on its horizontal and vertical parts.*

Vector geometry (Extended)

A vector can be drawn as a **directed line segment** 有向线段 (an arrow). The **position vector** 位置向量 of a point is the vector from the origin O to that point.

A key idea: the vector from A to B is

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a},$$

where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B.

Two vectors are **parallel** 平行 if one is a scalar multiple of the other (for example $\overrightarrow{AB} = 2\,\overrightarrow{CD}$). Three points are **collinear** 共线 (in a straight line) if the vectors between them are parallel and share a point. You can express any vector in terms of two **coplanar** 共面 vectors.

Worked example. O is the origin, with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of AB. Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

$$\overrightarrow{OM} = \mathbf{a} + \tfrac{1}{2}\overrightarrow{AB} = \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).$$